√SmarterMaths

Thank you for subscribing to SmarterMaths Teacher Edition (Silver) in 2018.

In order to continue to add value to our subscribers, we have put together our "HSC Comprehensive Revision Series" that we recommend motivated students aiming for a Band 5 or 6 result should ***attempt, carefully review and annotate*** in Term 3, creating a concise and high quality revision resource.

Our analysis on each topic, the common question types, past areas of difficulty and recent HSC trends all combine to create an extremely important revision set that ensures students cover a wide cross-section of the key areas we have carefully identified.

Note that a separate, more condensed "Final HSC Revision Set" for interested students will be available at the start of October in the final stretch before the Extension 1 HSC exam on 30 October, 2018.

IMPORTANT: If students have been exposed to many of the questions in these worksheets during the year, we say great! In sports vernacular, this is where cobwebs are turned into cables through repetition, confidence is built and speed through the paper is developed (an aspect we regard as critical to peak achievement).

HSC Final Study: EXT1 Topics 14 (~ 24% historical contribution)

Key Areas addressed by this worksheet

Topic 14: Other Motion (~ 2.5% historical contribution)

- covers motion outside of the major areas of projectile and simple harmonic motion;
- good mark allocations in 3 of the last 4 years;
- harder example from 2015 that caused problems reviewed.

Topic 14: Rates of Change (~ 4% historical contribution)

- *key revision focus* - asked in 9 of the last 10 years; 2017 exam's 1-mark allocation an outlier with previous questions all worth between 3-10 marks;
- underlying themes: area/volume questions, Pythagoras and non-right angled trig;
- constant rate of change (caused problems in the past).

Topic 14: Exponential Growth and Decay (~ 3.5% historical contribution)

- asked in 7 of the last 8 years; highest mean marks of any topic 14 sub-category;
- standard proof of rate of growth differential equation (regularly asked in early part of questions);
- 2017 question caused major problems and is reviewed.

Extension 1 Mathematics
HSC Comprehensive Revision Series

14. Calculus and the Physical World EXT1
Exponential Growth and Decay EXT1
Other Motion EXT1
Rates of Change EXT1

Teacher: SmarterMaths
Exam Equivalent Time: 75 minutes *(based on HSC allocation of 1.5 minutes approx. per mark)*

Questions

1. Calculus in the Physical World, EXT1 2015 HSC 2 MC

Given that $N = 100 + 80e^{kt}$, which expression is equal to $\dfrac{dN}{dt}$?

(A) $k(100 - N)$

(B) $k(180 - N)$

(C) $k(N - 100)$

(D) $k(N - 180)$

2. Calculus in the Physical World, EXT1 2017 HSC 8 MC

A stone drops into a pond, creating a circular ripple. The radius of the ripple increases from 0 cm, at a constant rate of $5\ \text{cm s}^{-1}$.

At what rate is the area enclosed within the ripple increasing when the radius is 15 cm?

A. $25\pi\ \text{cm}^2\ \text{s}^{-1}$

B. $30\pi\ \text{cm}^2\ \text{s}^{-1}$

C. $150\pi\ \text{cm}^2\ \text{s}^{-1}$

D. $225\pi\ \text{cm}^2\ \text{s}^{-1}$

3. Calculus in the Physical World, EXT1 2016 HSC 7 MC

The displacement x of a particle at time t is given by

$$x = 5\sin 4t + 12\cos 4t.$$

What is the maximum velocity of the particle?

(A) 13

(B) 28

(C) 52

(D) 68

4. Calculus in the Physical World, EXT1 2010 HSC 2b

The mass M of a whale is modelled by

$$M = 36 - 35.5e^{-kt}$$

where M is measured in tonnes, t is the age of the whale in years and k is a positive constant.

(i) Show that the rate of growth of the mass of the whale is given by the differential equation

$$\frac{dM}{dt} = k(36 - M)$$ *(1 mark)*

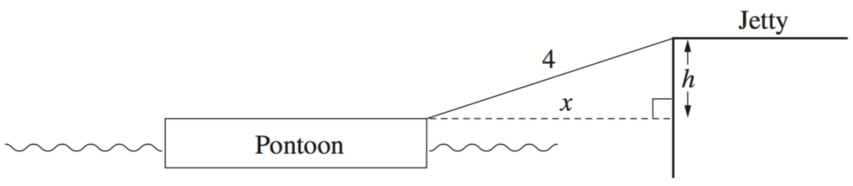
(ii) When the whale is 10 years old its mass is 20 tonnes.

Find the value of k, correct to three decimal places. *(2 marks)*

(iii) According to this model, what is the limiting mass of the whale? *(1 mark)*

5. Calculus in the Physical World, EXT1 2004 HSC 3c

A ferry wharf consists of a floating pontoon linked to a jetty by a 4 metre long walkway. Let h metres be the difference in height between the top of the pontoon and the top of the jetty and let x metres be the horizontal distance between the pontoon and the jetty.



(i) Find an expression for x in terms of h. *(1 mark)*

(ii) When the top of the pontoon is 1 metre lower than the top of the jetty, the tide is rising at a rate of 0.3 metres per hour.
At what rate is the pontoon moving away from the jetty? *(3 marks)*

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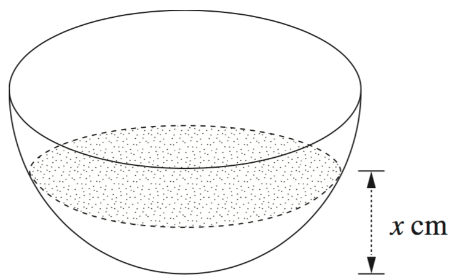
6. Calculus in the Physical World, EXT1 2006 HSC 5c



A hemispherical bowl of radius r cm is initially empty. Water is poured into it at a constant rate of k cm³ per minute. When the depth of water in the bowl is x cm, the volume, V cm³, of water in the bowl is given by

$$V = \frac{\pi}{3}x^2(3r - x). \qquad \text{(Do NOT prove this)}$$

(i) Show that

$$\frac{dx}{dt} = \frac{k}{\pi x(2r - x)}. \quad \textit{(2 marks)}$$

(ii) Hence, or otherwise, show that it takes 3.5 times as long to fill the bowl to the point where $x = \frac{2}{3}r$ as it does to fill the bowl to the point where $x = \frac{1}{3}r$. *(2 marks)*

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7. Calculus in the Physical World, EXT1 2013 HSC 13a

A spherical raindrop of radius r metres loses water through evaporation at a rate that depends on its surface area. The rate of change of the volume V of the raindrop is given by

$$\frac{dV}{dt} = -10^{-4}A,$$

where t is time in seconds and A is the surface area of the raindrop. The surface area and the volume of the raindrop are given by $A = 4\pi r^2$ and $V = \frac{4}{3}\pi r^3$ respectively.

(i) Show that $\frac{dr}{dt}$ is constant. *(1 mark)*

(ii) How long does it take for a raindrop of volume 10^{-6} m^3 to completely evaporate? *(2 marks)*

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8. Calculus in the Physical World, EXT1 2004 HSC 5a

A particle is moving along the x-axis, starting from a position 2 metres to the right of the origin (that is, $x = 2$ when $t = 0$) with an initial velocity of $5\ \text{ms}^{-1}$ and an acceleration given by

$$\ddot{x} = 2x^3 + 2x.$$

(i) Show that $\dot{x} = x^2 + 1$. *(2 marks)*

(ii) Hence find an expression for x in terms of t. *(3 marks)*

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9. Calculus in the Physical World, EXT1 2015 HSC 14b

A particle is moving horizontally. Initially the particle is at the origin O moving with velocity 1 m s $^{-1}$.

The acceleration of the particle is given by $\ddot{x} = x - 1$, where x is its displacement at time t.

(i) Show that the velocity of the particle is given by $\dot{x} = 1 - x$. *(3 marks)*

(ii) Find an expression for x as a function of t. *(2 marks)*

(iii) Find the limiting position of the particle. *(1 mark)*

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10. Calculus in the Physical World, EXT1 2016 HSC 12b

In a chemical reaction, a compound X is formed from a compound Y. The mass in grams of X and Y are $x(t)$ and $y(t)$ respectively, where t is the time in seconds after the start of the chemical reaction.

Throughout the reaction the sum of the two masses is 500 g. At any time t, the rate at which the mass of compound X is increasing is proportional to the mass of compound Y.

At the start of the chemical reaction, $x = 0$ and $\frac{dx}{dt} = 2$.

i. Show that $\frac{dx}{dt} = 0.004(500 - x)$. *(3 marks)*

ii. Show that $x = 500 - Ae^{-0.004t}$ satisfies the equation in part (i), and find the value of A. *(2 marks)*

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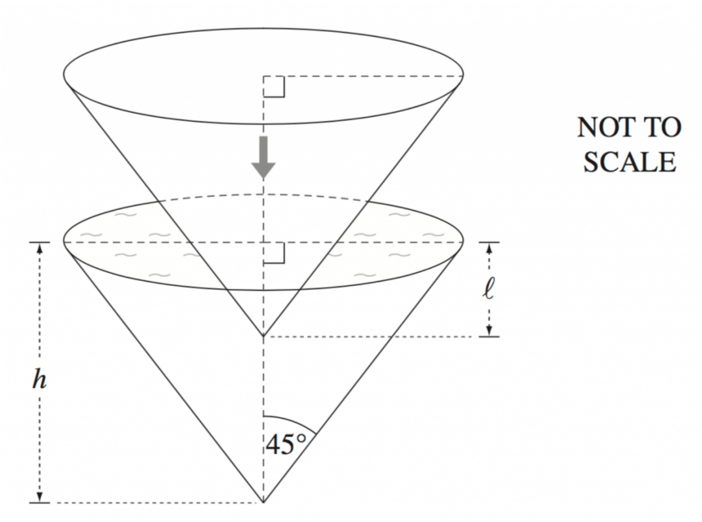
11. Calculus in the Physical World, EXT1 2011 HSC 7a

The diagram shows two identical circular cones with a common vertical axis. Each cone has height h cm and semi-vertical angle $45°$.



The lower cone is completely filled with water. The upper cone is lowered vertically into the water as shown in the diagram. The rate at which it is lowered is given by

$$\frac{dl}{dt} = 10,$$

where l cm is the distance the upper cone has descended into the water after t seconds.

As the upper cone is lowered, water spills from the lower cone. The volume of water remaining in the lower cone at time t is V cm³.

(i) Show that

$$V = \frac{\pi}{3}\left(h^3 - l^3\right).$$ *(1 mark)*

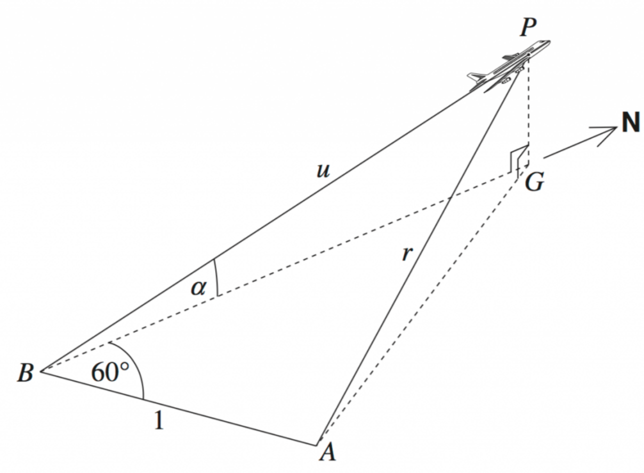
(ii) Find the rate at which V is changing with respect to time when $l = 2$. *(2 marks)*

(iii) Find the rate at which V is changing with respect to time when the lower cone has lost $\frac{1}{8}$ of its water. Give your answer in terms of h. *(2 marks)*

12. Calculus in the Physical World, EXT1 2012 HSC 14c

A plane P takes off from a point B. It flies due north at a constant angle α to the horizontal. An observer is located at A, 1 km from B, at a bearing $060°$ from B.

Let u km be the distance from B to the plane and let r km be the distance from the observer to the plane. The point G is on the ground directly below the plane.



(i) Show that $r = \sqrt{1 + u^2 - u\cos\alpha}$. *(3 marks)*

(ii) The plane is travelling at a constant speed of 360km/h.

At what rate, in terms of α, is the distance of the plane from the observer changing 5 minutes after take-off? *(2 marks)*

13. Calculus in the Physical World, EXT1 2017 HSC 14c

The concentration of a drug in a body is $F(t)$, where t is the time in hours after the drug is taken.

Initially the concentration of the drug is zero. The rate of change of concentration of the drug is given by

$$F'(t) = 50e^{-0.5t} - 0.4F(t).$$

(i) By differentiating the product $F(t)e^{0.4t}$ show that

$$\frac{d}{dt}\left(F(t)e^{0.4t}\right) = 50e^{-0.1t}.$$ *(2 marks)*

(ii) Hence, or otherwise, show that $F(t) = 500\left(e^{-0.4t} - e^{-0.5t}\right)$. *(2 marks)*

(iii) The concentration of the drug increases to a maximum.

For what value of t does this maximum occur? *(2 marks)*

Worked Solutions

1. Calculus in the Physical World, EXT1 2015 HSC 2 MC

$$N = 100 + 80e^{kt}$$

$$\frac{dN}{dt} = k \times 80e^{kt}$$

$$= k(N - 100)$$

$$\Rightarrow C$$

2. Calculus in the Physical World, EXT1 2017 HSC 8 MC

$$\frac{dr}{dt} = 5 \text{ cm}^2 \text{ s}^{-1}$$

$$A = \pi r^2$$

$$\frac{dA}{dr} = 2\pi r$$

$$\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}$$

$$= 2\pi r \cdot 5$$

$$= 10\pi r$$

When $r = 15$

$$\frac{dA}{dr} = 10\pi \cdot 15$$

$$= 150\pi \text{ cm}^2 \text{ s}^{-1}$$

$$\Rightarrow C$$

Worked Solutions

3. Calculus in the Physical World, EXT1 2016 HSC 7 MC

$x = 5\sin 4t + 12\cos 4t$

$\dfrac{dx}{dt} = 20\cos 4t - 48\sin 4t$

$\Rightarrow$ Can be written in the form:

$A\cos(4t + \alpha)$, where

$A = \sqrt{20^2 + 48^2}$

$= 52$

$\therefore$ Max $v = 52\ \text{ms}^{-1}$

$\Rightarrow$ C

4. Calculus in the Physical World, EXT1 2010 HSC 2b

(i) $M = 36 - 35.5e^{-kt}$

$35.5e^{-kt} = 36 - M$

$\therefore \dfrac{dM}{dt} = -k \times -35.5e^{-kt}$

$= k \times 35.5e^{-kt}$

$= k(36 - M)$ ... as required

IMPORTANT: Students must be well practised in this standard proof and be able to produce it quickly.

(ii) Find k

When $t = 10$, $M = 20$

$M = 36 - 35.5e^{-kt}$

$20 = 36 - 35.5e^{-10k}$

$35.5e^{-10k} = 16$

$\ln e^{-10k} = \ln\left(\dfrac{16}{35.5}\right)$

$-10k = \ln\left(\dfrac{16}{35.5}\right)$

$\therefore k = -\dfrac{\ln\left(\frac{16}{35.5}\right)}{10}$

$= 0.07969...$

$= 0.080$ (to 3 d.p.)

(iii) As $t \to \infty$, $e^{-kt} = \dfrac{1}{e^{kt}} \to 0$, $k > 0$

$M \to 36$

$\therefore$ The whale's limiting mass is 36 tonnes.

5. Calculus in the Physical World, EXT1 2004 HSC 3c

(i) Using Pythagoras

$x^2 + h^2 = 4^2$

$$x^2 = 16 - h^2$$
$$x = \sqrt{16 - h^2}$$

(ii) Find $\dfrac{dx}{dt}$ when $h = 1$

$$\frac{dx}{dt} = \frac{dx}{dh} \cdot \frac{dh}{dt}$$
$$x = \left(16 - h^2\right)^{\frac{1}{2}}$$
$$\frac{dx}{dh} = \frac{1}{2} \times \left(16 - h^2\right)^{-\frac{1}{2}} \times \frac{d}{dh}\left(16 - h^2\right)$$
$$= \frac{1}{2}\left(16 - h^2\right)^{-\frac{1}{2}} \times -2h$$
$$= \frac{-h}{\sqrt{16 - h^2}}$$

When $h = 1$, $\dfrac{dh}{dt} = -0.3$ m/hr

(h decreases when the tide is rising)

$$\frac{dx}{dt} = \frac{-h}{\sqrt{16 - h^2}} \times -0.3$$
$$= \frac{-1}{\sqrt{16 - 1^2}} \times -0.3$$
$$= \frac{0.3}{\sqrt{15}}$$
$$= 0.0774\ldots$$
$$= 0.077 \quad \text{metres per hr (to 2 d.p.)}$$

$\therefore$ When $h = 1$, the pontoon is moving away at 0.077 metres per hr.

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6. Calculus in the Physical World, EXT1 2006 HSC 5c

(i) Show $\dfrac{dx}{dt} = \dfrac{k}{\pi x(2r - x)}$

$$\frac{dV}{dt} = k$$
$$V = \frac{\pi}{3}x^2(3r - x)$$
$$= r\pi x^2 - \frac{\pi}{3}x^3$$
$$\frac{dV}{dx} = 2\pi rx - \pi x^2$$
$$= \pi x(2r - x)$$

$$\frac{dV}{dt} = \frac{dV}{dx} \cdot \frac{dx}{dt}$$
$$k = \pi x(2r - x) \cdot \frac{dx}{dt}$$
$$\therefore \frac{dx}{dt} = \frac{k}{\pi x(2r - x)} \quad \ldots \text{ as required}$$

(ii) $$\frac{dx}{dt} = \frac{k}{\pi x(2r - x)}$$
$$\frac{dt}{dx} = \frac{1}{k}\pi x(2r - x)$$
$$t = \frac{1}{k}\int 2\pi rx - \pi x^2 \, dx$$
$$= \frac{1}{k}\left[\pi rx^2 - \frac{1}{3}\pi x^3\right] + c$$

When $t = 0$, $x = 0$

$\therefore c = 0$

$$\therefore t = \frac{1}{k}\left[\pi rx^2 - \frac{1}{3}\pi x^3\right]$$

Find t_1, when $x = \frac{1}{3}r$

$$t_1 = \frac{1}{k}\left[\pi r\left(\frac{r}{3}\right)^2 - \frac{1}{3}\pi\left(\frac{r}{3}\right)^3\right]$$

$$= \frac{1}{k}\left[\frac{\pi r^3}{9} - \frac{\pi r^3}{81}\right]$$

$$= \frac{1}{k}\left(\frac{9\pi r^3}{81} - \frac{\pi r^3}{81}\right)$$

$$= \frac{8\pi r^3}{81k}$$

Find t_2 when $x = \frac{2}{3}r$

$$t_2 = \frac{1}{k}\left[\pi r\left(\frac{2r}{3}\right)^2 - \frac{1}{3}\pi\left(\frac{2r}{3}\right)^3\right]$$

$$= \frac{1}{k}\left[\frac{4\pi r^3}{9} - \frac{8\pi r^3}{81}\right]$$

$$= \frac{1}{k}\left(\frac{36\pi r^3}{81} - \frac{8\pi r^3}{81}\right)$$

$$= \frac{28\pi r^3}{81k}$$

$$= 3.5 \times \frac{8\pi r^3}{81k}$$

$$= 3.5 \times t_1$$

$\therefore$ It takes 3.5 times longer to fill the bowl.

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7. Calculus in the Physical World, EXT1 2013 HSC 13a

(i) Need to show $\frac{dr}{dt}$ is a constant

$$\frac{dV}{dt} = \frac{dV}{dr}\cdot\frac{dr}{dt} \quad \text{... (1)}$$

$$V = \frac{4}{3}\pi r^3$$

$$\therefore \frac{dV}{dr} = 4\pi r^2$$

$$\frac{dV}{dt} = -10^{-4}A \ \text{(given)}$$

Substituting into (1)

$$-10^{-4}A = 4\pi r^2 \times \frac{dr}{dt}$$

$$= A \times \frac{dr}{dt}$$

$$\therefore \frac{dr}{dt} = -10^{-4} \quad \text{... as required}$$

(ii) Using $V = 10^{-6}$ m³

$$\frac{4}{3}\pi r^3 = 10^{-6}$$

$$r^3 = \frac{3 \times 10^{-6}}{4\pi}$$

$$r = \sqrt[3]{\frac{3 \times 10^{-6}}{4\pi}}$$

Since the radius decreases at a constant rate,

$$t = \frac{\sqrt[3]{\frac{3\times 10^{-6}}{4\pi}}}{10^{-4}}$$

$$= 62.035...$$

$$= 62 \text{ seconds (nearest whole)}$$

♦♦ Mean mark 31%

$\therefore$ It takes 62 seconds for then raindrop to evaporate.

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8. Calculus in the Physical World, EXT1 2004 HSC 5a

(i) Show $\dot{x} = x^2 + 1$

$$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = 2x^3 + 2x$$

$$\therefore \frac{1}{2}v^2 = \int 2x^3 + 2x\,dx$$

$$= \frac{2}{4}x^4 + x^2 + c$$

$$v^2 = x^4 + 2x^2 + c$$

When $x = 2, \ v = 5$

$$5^2 = 2^4 + \left(2 \times 2^2\right) + c$$

$$25 = 16 + 8 + c$$

$$c = 1$$

$$\therefore v^2 = x^4 + 2x^2 + 1$$

$$= \left(x^2 + 1\right)^2$$

$$v = \sqrt{\left(x^2 + 1\right)^2}$$

$\therefore \dot{x} = x^2 + 1$ ... as required

(ii) $\dfrac{dx}{dt} = x^2 + 1$

$$\frac{dt}{dx} = \frac{1}{x^2 + 1}$$

$$\therefore t = \int \frac{1}{x^2 + 1}\,dx$$

$$= \tan^{-1} x + c$$

When $t = 0, \ x = 2$

$$0 = \tan^{-1} 2 + c$$

$$c = -\tan^{-1} 2$$

$$\therefore t = \tan^{-1} x - \tan^{-1} 2$$

$$\tan^{-1} x = t + \tan^{-1} 2$$

$$\therefore x = \tan\left(t + \tan^{-1} 2\right)$$

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9. Calculus in the Physical World, EXT1 2015 HSC 14b

(i) $\ddot{x} = \dfrac{d}{dx}\left(\dfrac{1}{2}\dot{x}^2\right) = x - 1$

$$\frac{1}{2}\dot{x}^2 = \int \ddot{x}\,dx$$

$$= \int x - 1\,dx$$

$$= \frac{1}{2}x^2 - x + c$$

When $x = 0, \dot{x} = 1$

$$\frac{1}{2} \cdot 1^2 = 0 - 0 + c$$

$$c = \frac{1}{2}$$

$$\therefore \frac{1}{2}\dot{x}^2 = \frac{1}{2}x^2 - x + \frac{1}{2}$$

$$\dot{x}^2 = x^2 - 2x + 1$$

$$= (x - 1)^2$$

$$\therefore \dot{x} = \pm(x - 1)$$

Since $V = 1$ when $x = 0$,

$\dot{x} = 1 - x$ … as required

(ii) $\dfrac{dx}{dt} = 1 - x$ (from (i))

$$\frac{dt}{dx} = \frac{1}{1 - x}$$

$$t = \int \frac{1}{1-x}\,dx$$
$$= -\ln(1-x) + c$$

When $t = 0, x = 0$

$$0 = -\ln 1 + c$$
$$\therefore c = 0$$
$$t = -\ln(1-x)$$
$$-t = \ln(1-x)$$
$$1 - x = e^{-t}$$
$$\therefore x = 1 - e^{-t}$$

♦ *Mean mark 49%.*

(iii) As $t \to \infty$

$$e^{-t} \to 0$$
$$x \to 1$$

$\therefore$ Limiting position is $x = 1$

♦ *Mean mark 42%.*

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10. Calculus in the Physical World, EXT1 2016 HSC 12b

i. $x + y = 500$ (given)

$$\frac{dx}{dt} = ky$$
$$= k(500 - x)$$

When $t = 0,\ x = 0,\ \dfrac{dx}{dt} = 2$

$$2 = k(500 - 0)$$
$$\therefore k = 0.004$$

$$\therefore \frac{dx}{dt} = 0.004(500 - x) \ldots \text{ as required}$$

ii. $$x = 500 - Ae^{-0.004t}$$
$$Ae^{-0.004t} = 500 - x$$

$$\frac{dx}{dt} = 0.004Ae^{-0.004t}$$
$$= 0.004(500 - x)$$

When $t = 0,\ x = 0,$

$$0 = 500 - Ae^{0}$$
$$\therefore A = 500$$

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11. Calculus in the Physical World, EXT1 2011 HSC 7a

(i) Show that $V = \dfrac{\pi}{3}\left(h^3 - l^3\right)$

Since $\tan 45^\circ = \dfrac{r}{h} = 1$

$$\Rightarrow r = h$$

♦ *Mean mark 42%*

$\Rightarrow$ Radius of lower cone $= h$

$$\therefore\ V(\text{lower cone}) = \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{3}\pi h^3$$

Similarly

$$V(\text{submerged upper cone}) = \frac{1}{3}\pi l^3$$

$$V(\text{water left}) = \frac{1}{3}\pi h^3 - \frac{1}{3}\pi l^3$$
$$= \frac{\pi}{3}\left(h^3 - l^3\right) \quad \text{... as required}$$

(ii) Find $\dfrac{dV}{dt}$ at $l = 2$

$$\frac{dV}{dt} = \frac{dV}{dl} \times \frac{dl}{dt} \text{ ... (1)}$$

$$\Rightarrow \frac{dl}{dt} = 10 \text{ (given)}$$

Using $V = \dfrac{\pi}{3}\left(h^3 - l^3\right)$ from part (i)

$$\Rightarrow \frac{dV}{dl} = -3 \times \frac{\pi}{3} l^2$$
$$= -\pi l^2$$

At $l = 2$

Substitute into (1) above

$$\frac{dV}{dt} = -\pi \times 2^2 \times 10$$
$$= -40\pi \quad \text{cm}^3/\text{sec}$$

(iii) Find $\dfrac{dV}{dt}$ when lower cone has lost $\dfrac{1}{8}$

Find l when $V = \dfrac{7}{8} \times \dfrac{1}{3}\pi h^3$

♦♦♦ *Mean mark 12%*
MARKER'S COMMENT: Many unsuccessful answers attempted to find an alternate version of $\frac{dV}{dt}$. Part (ii) directed students directly toward the correct strategy.

$$\frac{\pi}{3}\left(h^3 - l^3\right) = \frac{7}{8} \times \frac{1}{3}\pi h^3$$
$$h^3 - l^3 = \frac{7}{8}h^3$$
$$l^3 = \frac{1}{8}h^3$$
$$l = \frac{h}{2}$$

When $l = \dfrac{h}{2}$

$$\frac{dV}{dt} = -\pi\left(\frac{h}{2}\right)^2 \times 10 \text{ ... (*)}$$
$$= \frac{-5\pi h^2}{2} \text{ cm}^3/\text{sec}$$

$\therefore$ V is decreasing at the rate of $\dfrac{5\pi h^2}{2}$ cm³/sec.

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12. Calculus in the Physical World, EXT1 2012 HSC 14c

(i) Show $r = \sqrt{1 + u^2 - u\cos\alpha}$

In ΔPGB

$$\cos\alpha = \frac{BG}{u}$$
$$BG = u\cos\alpha$$
$$\sin\alpha = \frac{PG}{u}$$
$$PG = u\sin\alpha$$

♦ *Mean mark 42%*
IMPORTANT: Students should always be looking for opportunities to use the identity $\sin^2\alpha + \cos^2\alpha = 1$ to clean up calculations with trig functions.

In ΔPGA, using Pythagoras

$$AG^2 = r^2 - PG^2$$
$$= r^2 - u^2\sin^2\alpha$$

Now using cosine rule in ΔABG

$$AG^2 = BG^2 + AB^2 - 2 \times BG \times AB \times \cos 60^\circ$$

$$r^2 - u^2 \sin^2 \alpha = u^2 \cos^2 \alpha + 1 - 2(u \cos \alpha) \times 1 \times \frac{1}{2}$$

$$\begin{aligned} r^2 &= u^2 \cos^2 \alpha + u^2 \sin^2 \alpha + 1 - u \cos \alpha \\ &= u^2 \left(\cos^2 \alpha + \sin^2 \alpha\right) + 1 - u \cos \alpha \\ &= u^2 + 1 - u \cos \alpha \\ r &= \sqrt{u^2 + 1 - u \cos \alpha} \text{ ... as required} \end{aligned}$$

(ii) Need to find $\dfrac{dr}{dt}$ when $t = 5$

♦♦♦ Mean mark 14%

$$\frac{dr}{dt} = \frac{dr}{du} \times \frac{du}{dt}$$

$$r = \left(u^2 + 1 - u \cos \alpha\right)^{\frac{1}{2}}$$

$$\begin{aligned} \frac{dr}{du} &= \frac{1}{2}\left(u^2 + 1 - u \cos \alpha\right)^{-\frac{1}{2}} \times (2u - \cos \alpha) \\ &= \frac{2u - \cos \alpha}{2\sqrt{u^2 + 1 - u \cos \alpha}} \end{aligned}$$

$$\frac{du}{dt} = 360 \text{ km/hr} \quad \text{(plane's speed)}$$

After 5 mins

$$u = \frac{5}{60} \times 360 = 30 \text{ km}$$

$$\begin{aligned} \frac{dr}{dt} &= \frac{(2 \times 30) - \cos \alpha}{2\sqrt{30^2 + 1 - 30 \cos \alpha}} \times 360 \\ &= \frac{180(60 - \cos \alpha)}{\sqrt{901 - 30 \cos \alpha}} \text{ km/hr} \end{aligned}$$

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13. Calculus in the Physical World, EXT1 2017 HSC 14c

(i) Show $\dfrac{d}{dt}\left(F(t)e^{0.4t}\right) = 50e^{-0.1t}$

$$F'(t) = 50e^{-0.5t} - 0.4F(t) \ldots (1) \quad \text{(given)}$$

Using product rule:

$$\begin{aligned} \frac{d}{dt}\left(F(t)e^{0.4t}\right) &= F'(t)e^{0.4t} + 0.4e^{0.4t}F(t) \\ &= e^{0.4t}(F'(t) + 0.4F(t)) \\ &= e^{0.4t} \cdot 50e^{-0.5t} \quad \text{(using (1) above)} \\ &= 50e^{-0.1t} \quad \ldots \text{ as required} \end{aligned}$$

(ii) Show $F(t) = 500\left(e^{-0.4t} - e^{-0.5t}\right)$

♦♦ Mean mark 37%.

$$\begin{aligned} F(t)e^{0.4t} &= \int 50e^{-0.1t}\, dt \\ &= \frac{50}{-0.1} \cdot e^{-0.1t} + c \\ &= -500e^{-0.1t} + c \end{aligned}$$

When $t = 0$, $F(t) = 0$

$$\begin{aligned} 0 &= -500e^0 + c \\ c &= 500 \end{aligned}$$

$$\begin{aligned} F(t)e^{0.4t} &= 500 - 500e^{-0.1t} \\ \therefore F(t) &= 500e^{-0.4t} - 500e^{-0.5t} \\ &= 500\left(e^{-0.4t} - e^{-0.5t}\right) \end{aligned}$$

(iii) $F(t) = 500\left(e^{-0.4t} - e^{-0.5t}\right)$

$$F'(t) = 500\left(-0.4e^{-0.4t} + 0.5e^{-0.5t}\right)$$

Find t when $F(t) = 0$

$$\begin{aligned}
0.4e^{-0.4t} &= 0.5e^{-0.5t} \\
\frac{e^{-0.4t}}{e^{-0.5t}} &= \frac{0.5}{0.4} \\
e^{0.1t} &= \frac{5}{4} \\
0.1t &= \ln\left(\frac{5}{4}\right) \\
\therefore\ t_{\max} &= 10\ln\left(\frac{5}{4}\right)
\end{aligned}$$

♦♦ *Mean mark 39%.*