MATHEMATICS ADVANCED

SmarterEd Maths & Sciences

COMPREHENSIVE REVISION SERIES 2024

Thank you for subscribing to SmarterMaths Teacher Edition (Silver) in 2024.

Key features of the Advanced "2024 HSC Comprehensive Revision Series" for include:

- ~22 hours of cherry picked HSC revision questions by topic
- Targeted at motivated students aiming for a Band 5 or 6 result
- Weighting toward more difficult examples
- Mark allocations given to each topic generally reflect its historical (new syllabus) HSC exam allocation.
- ***Attempt, carefully review and annotate*** this revision set in Term 3
- This question set provides the foundation of a concise and high quality revision resource for the run into the HSC exam.
- This resource should be used to complement (not replace) the critical final stretch preparation for every student - timed full exam practice papers.

Our analysis on each topic, the common question types, past areas of difficulty and recent HSC trends all combine to create this revision set that ensures students cover a wide cross-section of the key areas.

IMPORTANT: If students have been exposed to questions in these worksheets during the year, we say great. Many top performing students attest to the benefits of doing quality questions 2-3 times before the HSC. This type of revision set is aimed at creating confidence and *speed through the exam*, with cherry picked questions that cover all important elements of revision while avoiding low percentage rabbit hole excursions.

HSC Final Study: C4 Integration (1 of 2)

(~14.0% contribution to new syllabus exams)

Key Areas addressed by this worksheet

Integrals (3.0%)

- "Integrals" is comprised of standard, log, exponential and trig integration and has been a healthy contributor since the new syllabus was introduced in 2020.
- Standard integration has been tested every year since 2021. We look at multiple examples including the challenging *2011 Q2e* and *2011 Q4d*.
- Log and exponential integrals are due after being omitted in the 2021-23 exams and multiple examples are reviewed.
- Trig integrals have likewise been absent in the 2021-23 period. Well covered with a weighting toward harder examples.
- 2-step integration problems that require differentiation then integration were examined in both 2022 and 2020. An important revision area that has proven challenging in the past.

Other Integration Applications (2.8%)

- A feast or famine topic area which attracted three separate questions in 2023, nothing in 2022 and an 8-mark cross topic question in the 2021 exam that combined integration applications with trig graphs.
- Questions that overlap with *C3 Rates of Change* feature in the revision set, including the most-examined themes of motion, flow and population.
- Questions that provide the gradient/primitive function of a curve and require integration are another focus area, including the challenging *2023 Q28*.

ADVANCED
Stage 6

2024 Comprehensive Revision Series - CALCULUS

C4 Integration (Y12) ... 1 of 2
Integrals
Other Integration Techniques

Exam Equivalent Time: 90 minutes *(based on allocation of 1.5 minutes per mark)*

Questions

1. Calculus, 2ADV C4 2020 HSC 4 MC

What is $\int e + e^{3x}\ dx$?

A. $ex + 3e^{3x} + c$

B. $ex + \frac{1}{3}e^{3x} + c$

C. $e + 3e^{3x} + c$

D. $e + \frac{1}{3}e^{3x} + c$

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2. Calculus, 2ADV C4 2022 HSC 6 MC

What is $\int \frac{1}{(2x+1)^2}\ dx$?

A. $\frac{-2}{2x+1} + C$

B. $\frac{-1}{2(2x+1)} + C$

C. $2\ln(2x+1) + C$

D. $\frac{1}{2}\ln(2x+1) + C$

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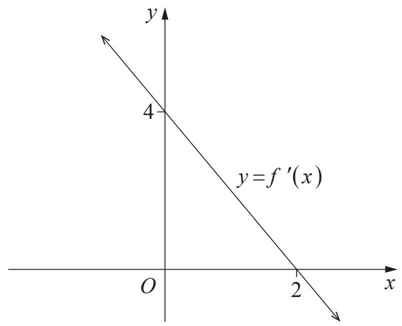
3. Calculus, 2ADV C3 2017 HSC 9 MC

The graph of $y = f'(x)$ is shown.



The curve $y = f(x)$ has a maximum value of 12.

What is the equation of the curve $y = f(x)$?

A. $y = x^2 - 4x + 12$

B. $y = 4 + 4x - x^2$

C. $y = 8 + 4x - x^2$

D. $y = x^2 - 4x + 16$

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4. Calculus, 2ADV C4 2019 HSC 9 MC

Which expression is equal to $\int \tan^2 x\ dx$?

A. $\tan x - x + C$

B. $\tan x - 1 + C$

C. $\frac{\tan^3 x^2}{6} + C$

D. $\frac{\tan^3 x}{3} + C$

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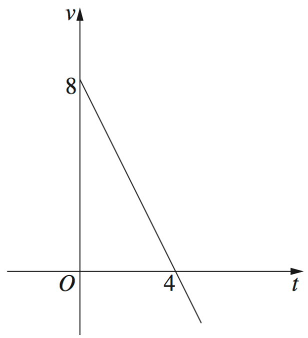
5. Calculus, 2ADV C4 2015 HSC 9 MC

A particle is moving along the x-axis. The graph shows its velocity v metres per second at time t seconds.



When $t = 0$ the displacement x is equal to 2 metres.

What is the maximum value of the displacement x?

A. 8 m

B. 14 m

C. 16 m

D. 18 m

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6. Calculus, 2ADV C4 2018 HSC 11e

Evaluate $\int_0^3 e^{5x}\, dx$. *(2 marks)*

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7. Calculus, 2ADV C4 2016 HSC 12d

i. Differentiate $y = xe^{3x}$. *(1 mark)*

ii. Hence find the exact value of $\int_0^2 e^{3x}(3+9x)\, dx$. *(2 marks)*

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8. Calculus, 2ADV C4 2022 HSC 18

a. Differentiate $y = \left(x^2+1\right)^4$. *(2 marks)*

b. Hence, or otherwise, find $\int x\left(x^2+1\right)^3 dx$. *(1 mark)*

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9. Calculus, 2ADV C4 2010 HSC 2e

Given that $\int_0^6 (x+k)\ dx = 30$, and k is a constant, find the value of k. *(2 marks)*

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10. Calculus, 2ADV C4 2006 HSC 2bii

Evaluate $\int_0^3 \frac{8x}{1+x^2}\ dx$. *(3 marks)*

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11. Calculus, 2ADV C4 2021 HSC 15

Evaluate $\int_{-2}^0 \sqrt{2x+4}\ dx$. *(2 marks)*

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12. Calculus, 2ADV C4 2023 HSC 13

Let $P(t)$ be a function such that $\frac{dP}{dt} = 3000e^{2t}$.

When $t = 0, P = 4000$.

Find an expression for $P(t)$. *(2 marks)*

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13. Calculus, 2ADV C4 2023 HSC 17

Find $\int x\sqrt{x^2+1}\ dx$ *(2 marks)*

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14. Calculus, 2ADV C4 2008 HSC 5a

The gradient of a curve is given by $\dfrac{dy}{dx} = 1 - 6\sin 3x$. The curve passes through the point $(0, 7)$.

What is the equation of the curve? *(3 marks)*

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15. Calculus, 2ADV C4 2010 HSC 2dii

Find $\displaystyle\int \frac{x}{4 + x^2}\ dx$. *(2 marks)*

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16. Calculus, 2ADV C4 2011 HSC 4b

Evaluate $\displaystyle\int_e^{e^3} \frac{5}{x}\ dx$ *(2 marks)*

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17. Calculus, 2ADV C4 2012 HSC 11g

Find $\displaystyle\int_0^{\frac{\pi}{2}} \sec^2\left(\frac{x}{2}\right)\ dx$ *(3 marks)*

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18. Calculus, 2ADV C4 2014 HSC 13a

i. Differentiate $3 + \sin 2x$. *(1 mark)*

ii. Hence, or otherwise, find $\displaystyle\int \frac{\cos 2x}{3 + \sin 2x}\ dx$. *(2 marks)*

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19. Calculus, 2ADV C4 2019 HSC 13c

i. Differentiate $(\ln x)^2$. *(2 marks)*

ii. Hence, or otherwise, find $\displaystyle\int \frac{\ln x}{x}\ dx$. *(1 mark)*

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20. Calculus, 2ADV C4 2012 HSC 15b

The velocity of a particle is given by

$$v = 1 - 2\cos t,$$

where x is the displacement in metres and t is the time in seconds. Initially the particle is 3 m to the right of the origin.

i. Find the initial velocity of the particle. *(1 mark)*

ii. Find the maximum velocity of the particle. *(1 mark)*

iii. Find the displacement, x, of the particle in terms of t. *(2 marks)*

iv. Find the position of the particle when it is at rest for the first time. *(2 marks)*

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21. Calculus, 2ADV C4 2011 HSC 4d

i. Differentiate $y = \sqrt{9 - x^2}$ with respect to x. *(2 marks)*

ii. Hence, or otherwise, find $\displaystyle\int \frac{6x}{\sqrt{9 - x^2}}\ dx$. *(2 marks)*

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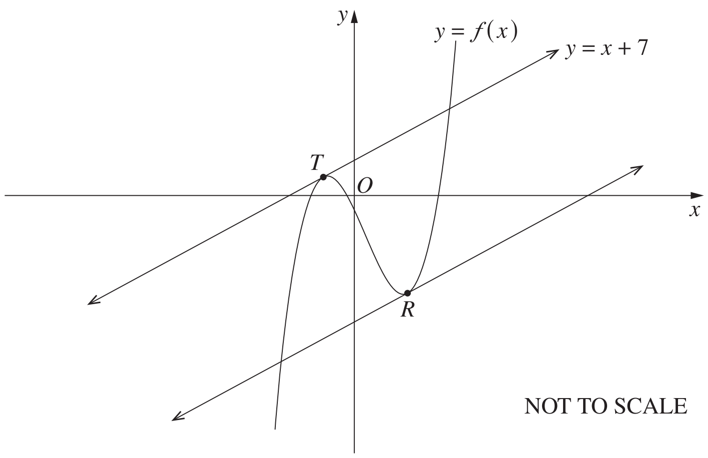
22. Calculus, 2ADV C4 2023 HSC 28

The curve $y = f(x)$ is shown on the diagram. The equation of the tangent to the curve at point $T(-1, 6)$ is $y = x + 7$. At a point R, another tangent parallel to the tangent at T is drawn.



The gradient function of the curve is given by $\dfrac{dy}{dx} = 3x^2 - 6x - 8$.

Find the coordinates of R. ***(4 marks)***

23. Calculus, 2ADV C4 2011 HSC 2e

Find $\displaystyle\int \frac{1}{3x^2}\,dx$. ***(2 marks)***

24. Calculus, 2ADV C4 2011 HSC 9b

A tap releases liquid A into a tank at the rate of $\left(2 + \dfrac{t^2}{t+1}\right)$ litres per minute, where t is time in minutes. A second tap releases liquid B into the same tank at the rate of $\left(1 + \dfrac{1}{t+1}\right)$ litres per minute. The taps are opened at the same time and release the liquids into an empty tank.

i. Show that the rate of flow of liquid A is greater than the rate of flow of liquid B by t litres per minute. ***(1 mark)***

ii. The taps are closed after 4 minutes. By how many litres is the volume of liquid A greater than the volume of liquid B in the tank when the taps are closed? ***(2 marks)***

Worked Solutions

Worked Solutions

1. Calculus, 2ADV C4 2020 HSC 4 MC

$$\int e + e^{3x} = ex + \frac{1}{3}e^{3x} + c$$

(Note e is a simple constant here)

$\Rightarrow B$

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2. Calculus, 2ADV C4 2022 HSC 6 MC

$$\int (2x+1)^{-2} = \frac{(2x+1)^{-1}}{(-1)(2)} + C$$

$$= \frac{-1}{2(2x+1)} + C$$

$\Rightarrow B$

Mean mark 53%.
COMMENT: A poor State result warrants attention.

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3. Calculus, 2ADV C3 2017 HSC 9 MC

Find the equation of $f'(x)$:

$m = -2, \ \ y\text{-int} = 4$

$y = -2x + 4$

$$f(x) = \int -2x + 4 \ dx$$

$$= -x^2 + 4x + c$$

Maximum $f(x) = 12$ when $f'(x) = 0$:

$-2x + 4 = 0$

$x = 2$

Substitute $x = 2$ into $f(x)$:

$\therefore 12 = -2^2 + 4 \cdot 2 + c$

$c = 8$

$\therefore f(x) = 8 + 4x - x^2$

$\Rightarrow C$

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4. Calculus, 2ADV C4 2019 HSC 9 MC

Consider option A:

$$\frac{d}{dx}(\tan x - x + C)$$

$$= \sec^2 x - 1$$

$$= \tan^2 x$$

$$\therefore \int \tan^2 x \ dx = \tan x - x + C$$

$\Rightarrow A$

♦♦ Mean mark 35%.

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5. Calculus, 2ADV C4 2015 HSC 9 MC

Distance travelled

$= \int_0^4 v\ dt$

♦♦ *Mean mark 31%.*

$=$Area under the velocity curve

$= \frac{1}{2} \times b \times h$

$= \frac{1}{2} \times 4 \times 8$

$= 16$ metres.

Since velocity is always positive between $t = 0$

and $t = 4$, and the original displacement $= 2$,

the maximum displacement $= 16 + 2 = 18$ metres

$\Rightarrow D$

6. Calculus, 2ADV C4 2018 HSC 11e

$$\int_0^3 e^{5x}\ dx = \left[\frac{1}{5}e^{5x}\right]_0^3$$

$$= \frac{1}{5}e^{15} - \frac{1}{5}e^0$$

$$= \frac{1}{5}\left(e^{15} - 1\right)$$

7. Calculus, 2ADV C4 2016 HSC 12d

i. $y = xe^{3x}$

Using product rule:

$$\frac{dy}{dx} = x \cdot 3e^{3x} + 1 \cdot e^{3x}$$

$$= e^{3x}(1 + 3x)$$

ii. $\int_0^2 e^{3x}(3 + 9x)\ dx$

$$= 3\int_0^2 e^{3x}(1 + 3x)\ dx$$

$$= 3\left[xe^{3x}\right]_0^2$$

$$= 3\left(2e^6 - 0\right)$$

$$= 6e^6$$

8. Calculus, 2ADV C4 2022 HSC 18

a. $y = \left(x^2 + 1\right)^4$

Using chain rule:

$$\frac{dy}{dx} = 4 \times 2x\left(x^2 + 1\right)^3$$

$$= 8x\left(x^2 + 1\right)^3$$

b. $\int x\left(x^2 + 1\right)^3\ dx = \frac{1}{8}\int 8x\left(x^2 + 1\right)^3\ dx$

$$= \frac{1}{8}\left(x^2 + 1\right)^4 + C$$

9. Calculus, 2ADV C4 2010 HSC 2e

$$\int_0^6 (x+k)\ dx = 30$$

$$\begin{aligned}\int_0^6 (x+k)\ dx &= \left[\frac{1}{2}\ x^2 + kx\right]_0^6 \\ &= \left[\left(\frac{1}{2} \times 6^2 + 6 \times k\right) - 0\right] \\ &= 18 + 6k\end{aligned}$$

$$\begin{aligned}\Rightarrow 18 + 6k &= 30 \\ 6k &= 12 \\ \therefore\ k &= 2\end{aligned}$$

10. Calculus, 2ADV C4 2006 HSC 2bii

$$\begin{aligned}&\int_0^3 \frac{8x}{1+x^2}\ dx \\ &= 4\int_0^3 \frac{2x}{1+x^2}\ dx \\ &= 4\left[\log_e\left(1+x^2\right)\right]_0^3 \\ &= 4[\log_e(1+9) - \log_e(1+0)] \\ &= 4[\log_e 10 - \log_e 1] \\ &= 4\log_e 10\end{aligned}$$

11. Calculus, 2ADV C4 2021 HSC 15

$$\begin{aligned}\int_{-2}^0 (2x+4)^{\frac{1}{2}}\ dx &= \left[\frac{2}{3} \cdot \frac{1}{2}(2x+4)^{\frac{3}{2}}\right]_{-2}^0 \\ &= \frac{1}{3}\left(4^{\frac{3}{2}} - 0\right) \\ &= \frac{8}{3}\end{aligned}$$

12. Calculus, 2ADV C4 2023 HSC 13

$$\begin{aligned}P(t) &= \int \frac{dP}{dt}\ dt \\ &= \int 3000e^{2t}\ dt \\ &= 1500e^{2t} + c\end{aligned}$$

When $t = 0, P = 4000$

$$\begin{aligned}4000 &= 1500e^0 + c \\ c &= 2500\end{aligned}$$

$$\therefore P(t) = 1500e^{2t} + 2500$$

13. Calculus, 2ADV C4 2023 HSC 17

Mean mark 52%.

$$\begin{aligned}&\int x\sqrt{x^2+1}\ dx \\ &= \frac{1}{2}\int 2x\left(x^2+1\right)^{\frac{1}{2}}\ dx \\ &= \frac{1}{2} \times \frac{2}{3}\left(x^2+1\right)^{\frac{3}{2}} + c \\ &= \frac{1}{3}\left(x^2+1\right)^{\frac{3}{2}} + c\end{aligned}$$

14. Calculus, 2ADV C4 2008 HSC 5a

$$\frac{dy}{dx} = 1 - 6\sin 3x$$

$$y = \int 1 - 6\sin 3x \ dx$$

$$= x + 2\cos 3x + c$$

Passes through $(0, 7)$:

$$\Rightarrow 0 + 2\cos 0 + c = 7$$

$$2 + c = 7$$

$$c = 5$$

$$\therefore \text{ Equation is } \quad y = x + 2\cos 3x + 5$$

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15. Calculus, 2ADV C4 2010 HSC 2dii

$$\int \frac{x}{4 + x^2} \ dx$$

$$= \frac{1}{2} \int \frac{2x}{4 + x^2} \ dx$$

$$= \frac{1}{2} \ln\left(4 + x^2\right) + C$$

IMPORTANT: Minimise errors by adjusting the integral to fit the form $\frac{f\prime(x)}{f(x)}$ *before* integrating.

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16. Calculus, 2ADV C4 2011 HSC 4b

$$\int_e^{e^3} \frac{5}{x} \ dx$$

$$= 5 \int_e^{e^3} \frac{1}{x} \ dx$$

$$= 5 [\ln x]_e^{e^3}$$

$$= 5\left(\ln e^3 - \ln e\right)$$

$$= 5(3 - 1)$$

$$= 10$$

MARKER'S COMMENT: Most common error was $\ln(5x)$. Minimize errors by getting the integral in the form of $\frac{f\prime(x)}{f(x)}$ *before* integrating.

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17. Calculus, 2ADV C4 2012 HSC 11g

$$\int_0^{\frac{\pi}{2}} \sec^2\left(\frac{x}{2}\right) \ dx$$

$$= \left[2\tan\left(\frac{x}{2}\right)\right]_0^{\frac{\pi}{2}}$$

$$= 2\tan \frac{\pi}{4} - 2\tan 0$$

$$= 2(1) - 0$$

$$= 2$$

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18. Calculus, 2ADV C4 2014 HSC 13a

i.

$$y = 3 + \sin 2x$$

$$\frac{dy}{dx} = 2\cos 2x$$

ii.

$$\int \frac{\cos 2x}{3 + \sin 2x} \ dx$$

$$= \frac{1}{2} \int \frac{2\cos 2x}{3 + \sin 2x} \ dx$$

$$= \frac{1}{2} \ln(3 + \sin 2x) + C \qquad \text{(from part (i))}$$

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19. Calculus, 2ADV C4 2019 HSC 13c

i. $y = (\ln x)^2$

$$\frac{dy}{dx} = 2 \cdot \frac{1}{x} \cdot \ln x$$

$$= \frac{2\ln x}{x}$$

ii. $\displaystyle\int \frac{\ln x}{x}\, dx = \frac{1}{2}\int \frac{2\ln x}{x} dx$

$$= \frac{1}{2}(\ln x)^2 + C$$

♦ *Mean mark part (ii) 49%.*

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20. Calculus, 2ADV C4 2012 HSC 15b

i. Find v when $t = 0$:

$$v = 1 - 2\cos 0$$
$$= 1 - 2$$
$$= -1$$

$\therefore$ Initial velocity is -1 m/s.

ii. Solution 1

Max velocity occurs when $a = \dfrac{dv}{dt} = 0$

$$a = 2\sin t$$

Find t when $a = 0$:

$$2\sin t = 0$$
$$t = 0,\ \pi,\ 2\pi, \ldots$$

At $t = 0$, $v = -1$ m/s

At $t = \pi$, $v = 1 - 2(-1) = 3$ m/s

$\therefore$ Maximum velocity is 3 m/s

♦♦ *Mean mark 29%*
MARKER'S COMMENT: Solution 2 is more efficient here. Using the -1 and +1 limits of trig functions can be very a effective way to calculate max/min values.

Solution 2

$$v = 1 - 2\cos t$$

Since $-1 < \cos t < 1$

$$-2 < 2\cos t < 2$$
$$-1 < 1 - 2\cos t < 3$$

$\therefore$ Maximum velocity is 3 m/s

iii. $x = \displaystyle\int v\ dt$

$$= \int (1 - 2\cos t)\ dt$$

$= t - 2\sin t + c$

When $t = 0, \quad x = 3 \quad \text{(given)}$

$3 = 0 - 2\sin 0 + 3$

$c = 3$

$\therefore x = t - 2\sin t + 3$

iv. Find x when $v = 0$ (first time):

When $v = 0,$

$$0 = 1 - 2\cos t$$

$$\cos t = \frac{1}{2}$$

$$t = \cos^{-1}\left(\frac{1}{2}\right)$$

$$= \frac{\pi}{3} \quad \text{(first time)}$$

♦ *Mean mark 50%*
MARKER'S COMMENT: Many students found $t = \frac{\pi}{3}$ but failed to gain full marks by omitting to find x. Remember that for calculus, angles are measured in radians, NOT degrees!

Find x when $t = \frac{\pi}{3}$:

$$x = \frac{\pi}{3} - 2\sin\left(\frac{\pi}{3}\right) + 3$$

$$= \frac{\pi}{3} - 2 \times \frac{\sqrt{3}}{2} + 3$$

$$= \frac{\pi}{3} - \sqrt{3} + 3 \text{ units}$$

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21. Calculus, 2ADV C4 2011 HSC 4d

i. $y = \sqrt{9 - x^2}$

$$= \left(9 - x^2\right)^{\frac{1}{2}}$$

IMPORTANT: Some students might find calculations easier by rewriting the equation as $y = \left(9 - x^2\right)^{\frac{1}{2}}$.

$$\frac{dy}{dx} = \frac{1}{2} \times \left(9 - x^2\right)^{-\frac{1}{2}} \times \frac{d}{dx}\left(9 - x^2\right)$$

$$= \frac{1}{2} \times \left(9 - x^2\right)^{-\frac{1}{2}} \times -2x$$

$$= -\frac{x}{\sqrt{9 - x^2}}$$

ii. $\displaystyle\int \frac{6x}{\sqrt{9 - x^2}}\, dx = -6 \int \frac{-x}{\sqrt{9 - x^2}}\, dx$

$$= -6\left(\sqrt{9 - x^2}\right) + C$$

$$= -6\sqrt{9 - x^2} + C$$

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22. Calculus, 2ADV C4 2023 HSC 28

Gradient at $R = 1$:

♦ Mean mark 51%.

$$3x^2 - 6x - 8 = 1$$
$$3x^2 - 6x - 9 = 0$$
$$3(x^2 - 2x - 3 = 0$$
$$3(x+1)(x-3) = 0$$

x-coordinate of $R = 3$

$$y = \int 3x^2 - 6x - 8\, dx$$
$$= x^3 - 3x^2 - 8x + c$$

Graph passes through $(-1, 6)$:

$$6 = -1 - 3 + 8 + c$$
$$c = 2$$

$$y = x^3 - 3x^2 - 8x + 2$$

When $x = 3$:

$$y = 27 - 27 - 24 + 2 = -22$$

$$\therefore R(3, -22)$$

23. Calculus, 2ADV C4 2011 HSC 2e

$$\int \frac{1}{3x^2}\, dx = \frac{1}{3}\int x^{-2}\, dx$$
$$= \frac{1}{3} \times \frac{1}{-1} \times x^{-1} + C$$
$$= -\frac{1}{3x} + C$$

♦ *Mean mark 46%.*
MARKER'S COMMENT: Students who took the $\frac{1}{3}$ out the front before integrating made less errors, as illustrated in the Worked Solution.

24. Calculus, 2ADV C4 2011 HSC 9b

i. Show difference in flow rate $(D) = t$

♦♦♦ *Mean mark 16%*
MARKER'S COMMENT: Many students incorrectly differentiated in part (i). The 1 mark allocation indicates the answer will not require an involved multi-step process.

$$D = \left(2 + \frac{t^2}{t+1}\right) - \left(1 + \frac{1}{t+1}\right)$$
$$= \frac{2(t+1) + t^2}{t+1} - \frac{(t+1)+1}{t+1}$$
$$= \frac{2t + 2 + t^2 - t - 2}{t+1}$$
$$= \frac{t^2 + t}{t+1}$$
$$= \frac{t(t+1)}{t+1}$$
$$= t \quad \ldots \text{ as required}$$

ii. Difference in Volume

♦♦♦ *Mean mark 15%*
MARKER'S COMMENT: Few students were able to answer this part. Previous parts of any question should be front and centre of your thinking when working out strategies.

$$= \int_0^4 \left(2 + \frac{t^2}{1+t}\right) dt - \int_0^4 \left(1 + \frac{t}{1+t}\right) dt$$
$$= \int_0^4 t\, dt \qquad \text{(using part(i))}$$
$$= \left[\frac{t^2}{2}\right]_0^4$$
$$= \frac{16}{2} - 0$$
$$= 8$$

$\therefore$ There is 8 litres more of liquid A than B.