MATHEMATICS EXTENSION 1

SmarterEd Maths & Sciences

COMPREHENSIVE REVISION SERIES 2024

Thank you for subscribing to SmarterMaths Teacher Edition (Silver) in 2024.

Key features of the Extension 1 “2024 HSC Comprehensive Revision Series” include:

- ~18 hours of cherry-picked HSC revision questions by topic
- Targeted at motivated students aiming for a Band 5 or 6 result
- Weighting toward more difficult examples
- Mark allocations given to each topic generally reflect its historical (new syllabus) HSC exam allocation.
- ***Attempt, carefully review and annotate*** this revision set in Term 3
- This question set provides the foundation of a concise and high-quality revision resource for the run into the HSC exam.
- This resource should be used to complement (not replace) the critical final stretch preparation for every student – completing full practice papers in exam conditions.

Our analysis on each topic, the common question types, past areas of difficulty and recent HSC trends all combine to create this revision set that ensures students cover a wide cross-section of the key areas.

IMPORTANT: If students have been exposed to questions in these worksheets during the year, we say great. Many top performing students attest to the benefits of doing quality questions 2-3 times before the HSC. This type of revision set is aimed at creating confidence and *speed through the exam*, with cherry picked questions that cover all important elements of revision while avoiding low percentage rabbit hole excursions.

HSC Final Study: C2 Further Calculus Skills (~13.1% historical contribution)

Key Areas addressed by this worksheet

Integration By Substitution (4.6%)

- *Integration By Substitution* has been a super consistent contributor to new syllabus Ext1 exams, attracting a dedicated longer answer question every year.
- Questions are typically in the band 3-4 range with 3-mark allocations except for 2020 where 5 marks was allocated to a *differentiate then integrate* question.
- Several low to mid-band difficulty questions are covered, including slightly more challenging examples where limits become inverted after substitution.

Inverse Function Calculus (7.1%)

- *Inverse Function Calculus* has meaningfully exceeded expectations in its contributions to new syllabus exams, accounting for two separate questions in every exam since 2020.
- Standard differentiation and integration of inverse trig functions is the most common question type and is well covered.
- *2020 Ext1 13c* combined inverse trig functions and the chain rule and caused problems – a great revision question warranting careful attention.
- *2023 Ext1 14a* is another "must review" question that required an explanation if an inverse function exists (mean mark 54%) and the use of a composite function for differentiation that a large majority could not answer.
- *2022 Ext1 12c* involved a tangent application to this topic area is an important revision covered. We note this question is the first time the “arcsin” terminology has appeared.
- Examiners have ratcheted up the difficulty level by requiring students to differentiate inverse trig equations (containing two inverse trig functions) and sketch the graph. We review this area with selected past HSC questions.

Harder Trig Calculus (1.4%)

- *Harder trig Calculus* was most recently examined in the 2020-21 exams (notably absent in the last two years).
- Multiple questions look at the core skill of competently and quickly using the identities related to cos(2θ) for solving calculus problems. An area regularly examined and prone to silly errors.

EXTENSION 1
Stage 6

Smarter Ed

2024 Comprehensive Revision Series - CALCULUS

C2 Further Calculus Skills (Y12)
Inverse Functions Calculus
Integration By Substitution
Harder Trig Calculus

Exam Equivalent Time: 120 minutes *(based on allocation of 1.5 minutes per mark)*

Questions

1. Calculus, EXT1 C2 2020 HSC 3 MC

Which of the following is an anti-derivative of $\dfrac{1}{4x^2+1}$?

A. $2\tan^{-1}\left(\dfrac{x}{2}\right)+c$

B. $\dfrac{1}{2}\tan^{-1}\left(\dfrac{x}{2}\right)+c$

C. $2\tan^{-1}(2x)+c$

D. $\dfrac{1}{2}\tan^{-1}(2x)+c$

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2. Calculus, EXT1 C2 2014 HSC 6 MC

What is the derivative of $3\sin^{-1}\dfrac{x}{2}$?

A. $\dfrac{6}{\sqrt{4-x^2}}$

B. $\dfrac{3}{\sqrt{4-x^2}}$

C. $\dfrac{3}{2\sqrt{4-x^2}}$

D. $\dfrac{3}{4\sqrt{4-x^2}}$

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3. Calculus, EXT1 C2 2015 HSC 7 MC

What is the value of k such that $\displaystyle\int_0^k \frac{1}{\sqrt{4-x^2}}\,dx=\frac{\pi}{3}$?

A. 1

B. $\sqrt{3}$

C. 2

D. $2\sqrt{3}$

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4. Calculus, EXT1 C2 2019 HSC 3 MC

What is the derivative of $\tan^{-1}\dfrac{x}{2}$?

A. $\dfrac{1}{2(4+x^2)}$

B. $\dfrac{1}{4+x^2}$

C. $\dfrac{2}{4+x^2}$

D. $\dfrac{4}{4+x^2}$

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5. Calculus, EXT1 C2 SM-Bank 1 MC

With a suitable substitution, $\displaystyle\int_1^5 (2x-1)\sqrt{2x+1}\,dx$ can be expressed as

A. $\displaystyle\frac{1}{2}\int_1^5\left(u^{\frac{3}{2}}+u^{\frac{1}{2}}\right)\,du$

B. $\displaystyle 2\int_3^{11}\left(u^{\frac{3}{2}}+u^{\frac{1}{2}}\right)\,du$

C. $\displaystyle 2\int_1^5\left(u^{\frac{3}{2}}-2u^{\frac{1}{2}}\right)\,du$

D. $\displaystyle\frac{1}{2}\int_3^{11}\left(u^{\frac{3}{2}}-2u^{\frac{1}{2}}\right)\,du$

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6. Calculus, EXT1 C2 2013 HSC 5 MC

Which integral is obtained when the substitution $u = 1 + 2x$ is applied to $\int x\sqrt{1+2x}\ dx$?

A. $\frac{1}{4}\int (u-1)\sqrt{u}\ du$

B. $\frac{1}{2}\int (u-1)\sqrt{u}\ du$

C. $\int (u-1)\sqrt{u}\ du$

D. $2\int (u-1)\sqrt{u}\ du$

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7. Calculus, EXT1 C2 2012 HSC 7 MC

Which expression is equal to $\int \sin^2 3x\ dx$?

A. $\frac{1}{2}\left(x - \frac{1}{3}\sin 3x\right) + C$

B. $\frac{1}{2}\left(x + \frac{1}{3}\sin 3x\right) + C$

C. $\frac{1}{2}\left(x - \frac{1}{6}\sin 6x\right) + C$

D. $\frac{1}{2}\left(x + \frac{1}{6}\sin 6x\right) + C$

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8. Calculus, EXT1 C2 2022 HSC 9 MC

A given function $f(x)$ has an inverse $f^{-1}(x)$.

The derivatives of $f(x)$ and $f^{-1}(x)$ exist for all real numbers x.

The graphs $y = f(x)$ and $y = f^{-1}(x)$ have at least one point of intersection.

Which statement is true for all points of intersection of these graphs?

A. All points of intersection lie on the line $y = x$.

B. None of the points of intersection lie on the line $y = x$.

C. At no point of intersection are the tangents to the graphs parallel.

D. At no point of intersection are the tangents to the graphs perpendicular.

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9. Calculus, EXT1 C2 2021 HSC 11f

Evaluate $\int_0^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\ dx$. *(2 marks)*

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10. Calculus, EXT1 C2 2023 HSC 12a

Evaluate $\int_3^4 (x+2)\sqrt{x-3}\ dx$ using the substitution $u = x - 3$. *(3 marks)*

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11. Calculus, EXT1 C2 2010 HSC 1e

Use the substitution $u = 1 - x$ to evaluate $\int_0^1 x\sqrt{1-x}\ dx$. *(3 marks)*

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12. Calculus, EXT1 C2 2011 HSC 1d

Using the substitution $u = \sqrt{x}$, evaluate $\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}\ dx$. *(3 marks)*

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13. Calculus, EXT1 C2 2015 HSC 11e

Use the substitution $u = 2x - 1$ to evaluate $\int_1^2 \frac{x}{(2x-1)^2}\ dx$. *(3 marks)*

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14. Calculus, EXT1 C2 2005 HSC 1a

Find $\int \frac{1}{x^2+49}\ dx$. *(1 mark)*

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15. Calculus, EXT1 C2 2008 HSC 1b

Differentiate $\cos^{-1}(3x)$ with respect to x. *(2 marks)*

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16. Calculus, EXT1 C2 2008 HSC 1c

Evaluate $\displaystyle\int_{-1}^{1} \frac{1}{\sqrt{4-x^2}}\,dx$. *(2 marks)*

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17. Calculus, EXT1 C2 2005 HSC 3b

i. By expanding the left-hand side, show that

$\sin(5x+4x)+\sin(5x-4x)=2\sin 5x\cos 4x$ *(1 mark)*

ii. Hence find $\displaystyle\int \sin 5x \cos 4x\ dx$. *(2 marks)*

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18. Calculus, EXT1 C2 2019 HSC 13a

Use the substitution $u=\cos^2 x$ to evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sin 2x}{4+\cos^2 x}\,dx$. *(3 marks)*

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19. Calculus, EXT1 C2 2020 HSC 12d

Find $\displaystyle\int_0^{\frac{\pi}{2}} \cos 5x \sin 3x\ dx$. *(3 marks)*

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20. Calculus, EXT1 C2 2020 HSC 13a

i. Find $\dfrac{d}{d\theta}\left(\sin^3\theta\right)$. *(1 mark)*

ii. Use the substitution $x=\tan\theta$ to evaluate $\displaystyle\int_0^1 \frac{x^2}{(1+x^2)^{\frac{5}{2}}}\,dx$. *(4 marks)*

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21. Calculus, EXT1 C2 2022 HSC 12c

Find the equation of the tangent to the curve $y=x\ \arctan(x)$ at the point with coordinates $\left(1,\dfrac{\pi}{4}\right)$. Give your answer in the form $y=mx+c$ *(3 marks)*

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22. Calculus, EXT1 C2 2016 HSC 11c

Differentiate $3\tan^{-1}(2x)$. *(2 marks)*

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23. Calculus, EXT1 C2 2009 HSC 1f

Using the substitution $u=x^3+1$, or otherwise, evaluate $\displaystyle\int_0^2 x^2 e^{x^3+1}\,dx$. *(3 marks)*

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24. Calculus, EXT1 C2 2007 HSC 1e

Use the substitution $u=25-x^2$ to evaluate $\displaystyle\int_3^4 \frac{2x}{\sqrt{25-x^2}}\,dx$. *(3 marks)*

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25. Calculus, EXT1 C2 2013 HSC 11b

Find $\displaystyle\int \frac{1}{\sqrt{49-4x^2}}\,dx$. *(2 marks)*

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26. Calculus, EXT1 C2 2013 HSC 11f

Use the substitution $u=e^{3x}$ to evaluate $\displaystyle\int_0^{\frac{1}{3}} \frac{e^{3x}}{e^{6x}+1}\,dx$. *(3 marks)*

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27. Calculus, EXT1 C2 2014 HSC 11d

Evaluate $\displaystyle\int_2^5 \frac{x}{\sqrt{x-1}}\,dx$ using the substitution $x=u^2+1$. *(3 marks)*

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28. Calculus, EXT1 C2 2018 HSC 12c

Let $f(x) = \sin^{-1}x + \cos^{-1}x$.

i. Show that $f\prime(x) = 0$ *(1 mark)*

ii. Hence, or otherwise, prove

$$\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}.$$ *(1 mark)*

iii. Hence, sketch

$$f(x) = \sin^{-1}x + \cos^{-1}x.$$ *(1 mark)*

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29. Calculus, EXT1 C2 2023 HSC 14a

Let $f(x) = 2x + \ln x$, for $x > 0$.

i. Explain why the inverse of $f(x)$ is a function. *(1 mark)*

ii. Let $g(x) = f^{-1}(x)$. By considering the value of $f(1)$, or otherwise, evaluate $g'(2)$. *(2 mark)*

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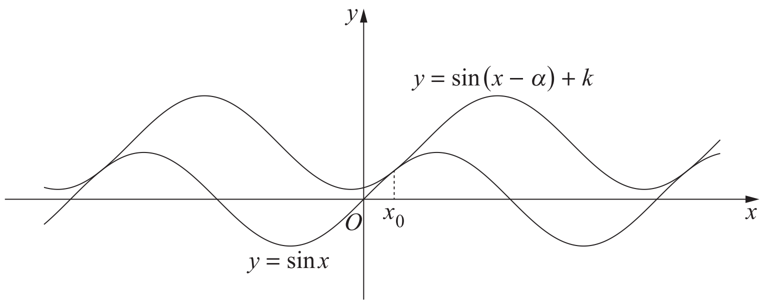
30. Calculus, EXT1 C2 2019 HSC 14c

The diagram shows the two curves $y = \sin x$ and $y = \sin(x - \alpha) + k$, where $0 < \alpha < \pi$ and $k > 0$. The two curves have a common tangent at x_0 where $0 < x_0 < \frac{\pi}{2}$.



i. Explain why $\cos x_0 = \cos(x_0 - \alpha)$. *(1 mark)*

ii. Show that $\sin x_0 = -\sin(x_0 - \alpha)$. *(2 marks)*

iii. Hence, or otherwise, find k in terms of α. *(2 marks)*

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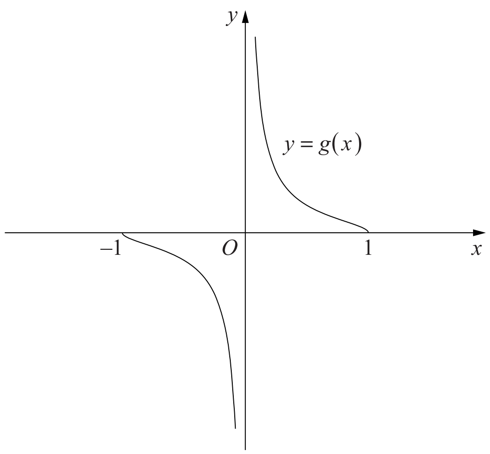
31. Calculus, EXT1 C2 2020 HSC 13c

Suppose $f(x) = \tan\left(\cos^{-1}(x)\right)$ and $g(x) = \frac{\sqrt{1-x^2}}{x}$.

The graph of $y = g(x)$ is given.



i. Show that $f'(x) = g'(x)$. *(4 marks)*

ii. Using part (i), or otherwise, show that $f(x) = g(x)$. *(3 marks)*

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32. Calculus, EXT1 C2 2021 HSC 14e

The polynomial $g(x) = x^3 + 4x - 2$ passes through the point (1, 3).

Find the gradient of the tangent to $f(x) = xg^{-1}(x)$ at the point where $x = 3$. *(2 marks)*

Worked Solutions

Worked Solutions

1. Calculus, EXT1 C2 2020 HSC 3 MC

$$\int \frac{1}{4x^2+1}\ dx = \frac{1}{2}\int \frac{2}{1+(2x)^2}\ dx$$
$$= \frac{1}{2}\tan^{-1}(2x)+c$$

$\Rightarrow D$

2. Calculus, EXT1 C2 2014 HSC 6 MC

$y = 3\sin^{-1}\ \frac{x}{2}$

$\frac{dy}{dx} = 3\times\frac{1}{\sqrt{4-x^2}}$

$\Rightarrow\ B$

3. Calculus, EXT1 C2 2015 HSC 7 MC

$$\int_0^1 \frac{1}{\sqrt{4-x^2}}dx = \frac{\pi}{3}$$
$$\left[\sin^{-1}\ \frac{x}{2}\right]_0^k = \frac{\pi}{3}$$
$$\sin^{-1}\ \frac{k}{2} - \sin^{-1}\ 0 = \frac{\pi}{3}$$
$$\sin^{-1}\ \frac{k}{2} = \frac{\pi}{3}$$
$$\frac{k}{2} = \sin\ \frac{\pi}{3}$$
$$= \frac{\sqrt{3}}{2}$$
$$\therefore k = \sqrt{3}$$

$\Rightarrow B$

4. Calculus, EXT1 C2 2019 HSC 3 MC

$y = \tan^{-1}\ \frac{x}{2}$

$$\frac{dy}{dx} = \frac{\frac{1}{2}}{1+\left(\frac{x}{2}\right)^2}$$
$$= \frac{1}{2\left(1+\frac{x^2}{4}\right)}$$
$$= \frac{2}{4+x^2}$$

$\Rightarrow\ C$

5. Calculus, EXT1 C2 SM-Bank 1 MC

Let $u = 2x+1\ \Rightarrow\ u-2 = 2x-1$

$\frac{du}{dx} = 2\ \Rightarrow\ dx = \frac{1}{2}du$

When $x = 5,\ u = 11$

When $x = 1,\ u = 3$

$$\therefore \int_1^5 (2x-1)\sqrt{2x+1}\ dx$$
$$= \frac{1}{2}\int_3^{11} (u-2)u^{\frac{1}{2}}\ du$$
$$= \frac{1}{2}\int_3^{11} u^{\frac{3}{2}}-2u^{\frac{1}{2}}\ du$$

$\Rightarrow D$

6. Calculus, EXT1 C2 2013 HSC 5 MC

Let $u = 1 + 2x$

$\therefore x = \frac{1}{2}(u - 1)$

$\frac{du}{dx} = 2$

$\therefore dx = \frac{1}{2}\ du$

$$\int x\sqrt{1+2x}\ dx$$

$$= \int \frac{1}{2}(u-1) \times u^{\frac{1}{2}} \times \frac{1}{2}\ du$$

$$= \frac{1}{4}\int (u-1)\sqrt{u}\ du$$

$\Rightarrow\ A$

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7. Calculus, EXT1 C2 2012 HSC 7 MC

Using: $\sin^2 a = \frac{1}{2}(1 - \cos 2a)$

$$\int \sin^2 3x\ dx = \frac{1}{2}\int (1 - \cos 6x)\ dx$$

$$= \frac{1}{2}\left(x - \frac{1}{6}\sin 6x\right) + C$$

$\Rightarrow\ C$

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8. Calculus, EXT1 C2 2022 HSC 9 MC

♦♦♦ Mean mark 13%.

By Elimination:

Consider $f(x) = x \quad \Rightarrow \quad f^{-1}(x) = x$:

All POI lie on $y = x$ and all tangents are parallel

→ Eliminate B and C

Consider $f(x) = -x \quad \Rightarrow \quad f^{-1}(x) = -x$:

All POI lie on $y = -x$

→ Eliminate A

$\Rightarrow D$

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9. Calculus, EXT1 C2 2021 HSC 11f

$$\int_0^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\ dx = \left[\sin^{-1}\frac{x}{2}\right]_0^{\sqrt{3}}$$

$$= \sin^{-1}\frac{\sqrt{3}}{2} - \sin^{-1}0$$

$$= \frac{\pi}{3}$$

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10. Calculus, EXT1 C2 2023 HSC 12a

$u = x - 3 \quad \Rightarrow \quad x = u + 3$

$\dfrac{du}{dx} = 1 \quad \Rightarrow \quad du = dx$

When $x = 4, u = 1$

When $x = 3, u = 0$

$$\begin{aligned} \int_3^4 (x+2)\sqrt{x-3}\,dx &= \int_0^1 (u+5)\sqrt{u}\,du \\ &= \int_0^1 u^{\frac{3}{2}} + 5u^{\frac{1}{2}}\,du \\ &= \left[\frac{2}{5} \times u^{\frac{5}{2}} + \frac{2}{3} \times 5u^{\frac{3}{2}}\right]_0^1 \\ &= \left[\left(\frac{2}{5} + \frac{10}{3}\right) - 0\right] \\ &= \frac{56}{15} \end{aligned}$$

11. Calculus, EXT1 C2 2010 HSC 1e

$u = 1 - x \quad \Rightarrow x = 1 - u$

$\dfrac{du}{dx} = -1 \quad \Rightarrow du = -dx$

When $x = 1, \quad u = 0$

$x = 0, \quad u = 1$

$$\begin{aligned} \therefore \int_0^1 x\sqrt{1-x}\ dx & \\ &= -\int_1^0 (1-u)u^{\frac{1}{2}}\ du \\ &= \int_1^0 (u-1)u^{\frac{1}{2}}\ du \\ &= \int_1^0 \left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right)\ du \\ &= \left[\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right]_1^0 \\ &= \left[0 - \left(\frac{2}{5} - \frac{2}{3}\right)\right] \\ &= -\left(\frac{6}{15} - \frac{10}{15}\right) \\ &= \frac{4}{15} \end{aligned}$$

12. Calculus, EXT1 C2 2011 HSC 1d

$$u = \sqrt{x} = x^{\frac{1}{2}}$$

$$\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

$$du = \frac{dx}{2\sqrt{x}}$$

$$\therefore 2du = \frac{dx}{\sqrt{x}}$$

When $x = 4, \quad u = 2$

$x = 1, \quad x = 1$

$$\therefore \int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}\ dx$$

$$= \int_1^2 e^u \times 2\ du$$

$$= 2[e^u]_1^2$$

$$= 2\left[e^2 - e^1\right]$$

$$= 2e(e-1)$$

13. Calculus, EXT1 C2 2015 HSC 11e

$$u = 2x - 1$$

$$\Rightarrow 2x = u + 1$$

$$x = \frac{1}{2}(u+1)$$

$$\frac{du}{dx} = 2$$

$$dx = \frac{du}{2}$$

When $x = 2,\ u = 3$

$x = 1,\ u = 1$

$$\therefore \int_1^2 \frac{x}{(2x-1)^2}\ dx$$

$$= \int_1^3 \frac{1}{2}(u+1) \cdot \frac{1}{u^2} \cdot \frac{du}{2}$$

$$= \frac{1}{4}\int_1^3 \left(\frac{u+1}{u^2}\right) du$$

$$= \frac{1}{4}\int_1^3 \frac{1}{u} + u^{-2} du$$

$$= \frac{1}{4}\left[\ln u - u^{-1}\right]_1^3$$

$$= \frac{1}{4}\left[\left(\ln 3 - \frac{1}{3}\right) - (\ln 1 - 1)\right]$$

$$= \frac{1}{4}\left(\ln 3 - \frac{1}{3} + 1\right)$$

$$= \frac{1}{4}\left(\ln 3 + \frac{2}{3}\right)$$

14. Calculus, EXT1 C2 2005 HSC 1a

$$\int \frac{1}{x^2 + 49}\ dx = \frac{1}{7}\int \frac{7}{x^2 + 7^2}\ dx$$

$$= \frac{1}{7}\tan^{-1}\frac{x}{7} + c$$

15. Calculus, EXT1 C2 2008 HSC 1b

$$y = \cos^{-1}(3x)$$

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-(3x)^2}} \cdot \frac{d}{dx}(3x)$$

$$= \frac{-3}{\sqrt{1-9x^2}}$$

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16. Calculus, EXT1 C2 2008 HSC 1c

$$\int_{-1}^{1} \frac{1}{\sqrt{4-x^2}}\ dx$$

$$= \left[\sin^{-1}\left(\frac{x}{2}\right)\right]_{-1}^{1}$$

$$= \sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}\left(-\frac{1}{2}\right)$$

$$= \frac{\pi}{6} - \left(-\frac{\pi}{6}\right)$$

$$= \frac{\pi}{3}$$

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17. Calculus, EXT1 C2 2005 HSC 3b

i. $\sin(5x+4x) + \sin(5x-4x) = 2\sin 5x \cos 4x$

$$\text{LHS} = \sin 5x \cos 4x - \sin 4x \cos 5x + \sin 5x \cos 4x$$

$$+ \sin 4x \cos 5x$$

$$= 2\sin 5x \cos 4x \ \ \ldots \text{ as required}$$

ii. $\int \sin 5x \cos 4x\ dx$

$$= \frac{1}{2}\int 2\sin 5x \cos 4x\ dx$$

$$= \frac{1}{2}\int \sin(5x+4x) + \sin(5x-4x)\ dx$$

$$= \frac{1}{2}\int \sin 9x + \sin x\ dx$$

$$= \frac{1}{2}\left[-\frac{1}{9}\cos 9x - \cos x\right] + c$$

$$= -\frac{1}{18}\cos 9x - \frac{1}{2}\cos x + c$$

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18. Calculus, EXT1 C2 2019 HSC 13a

$$u = \cos^2 x$$
$$\frac{du}{dx} = -2\sin x\cos x$$
$$= -\sin 2x$$
$$du = -\sin 2x\ dx$$

When $x = \frac{\pi}{4},\ \ u = \frac{1}{2}$

When $x = 0,\ \ u = 1$

$$\therefore \int_0^{\frac{\pi}{4}} \frac{\sin 2x}{4+\cos^2 x}\ dx = -\int_1^{\frac{1}{2}} \frac{du}{4+u}$$
$$= -\left[\ln(4+u)\right]_1^{\frac{1}{2}}$$
$$= -(\ln 4.5 - \ln 5)$$
$$= -\ln\frac{9}{10}$$
$$= \ln\frac{10}{9}$$

19. Calculus, EXT1 C2 2020 HSC 12d

$$\int_0^{\frac{\pi}{2}} \cos 5x \sin 3x\ dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} 2\cos 5x\sin 3x\ dx$$
$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin 8x - \sin 2x\ dx$$
$$= \frac{1}{2}\left[-\frac{1}{8}\cos 8x + \frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{2}}$$
$$= \frac{1}{2}\left[\left(-\frac{1}{8}\cos 4\pi + \frac{1}{2}\cos\pi\right) - \left(-\frac{1}{8}\cos 0 + \frac{1}{2}\cos 0\right)\right]$$
$$= \frac{1}{2}\left(-\frac{1}{8} - \frac{1}{2} + \frac{1}{8} - \frac{1}{2}\right)$$
$$= -\frac{1}{2}$$

20. Calculus, EXT1 C2 2020 HSC 13a

i. $\frac{d}{d\theta}\left(\sin^3\theta\right) = 3\cos\theta\sin^2\theta$

ii. Let $x = \tan\theta$

$$\frac{dx}{d\theta} = \sec^2\theta \Rightarrow dx = \sec^2\theta\ d\theta$$

When $x = 1,\ \theta = \frac{\pi}{4}$

When $x = 0,\ \theta = 0$

$$\int_0^1 \frac{x^2}{(1+x^2)^{\frac{5}{2}}}dx = \int_0^{\frac{\pi}{4}} \frac{\tan^2\theta}{\left(1+\tan^2\theta\right)^{\frac{5}{2}}} \times \sec^2\theta\ d\theta$$
$$= \int_0^{\frac{\pi}{4}} \frac{\tan^2\theta}{(\sec^2\theta)^{\frac{5}{2}}} \times \sec^2\theta\ d\theta$$
$$= \int_0^{\frac{\pi}{4}} \frac{\sin^2\theta}{\cos^2\theta} \cdot \frac{1}{(\sec^2\theta)^{\frac{3}{2}}}\ d\theta$$
$$= \int_0^{\frac{\pi}{4}} \frac{\sin^2\theta}{\cos^2\theta} \cdot \frac{1}{\sec^3\theta}\ d\theta$$
$$= \int_0^{\frac{\pi}{4}} \sin^2\theta\cos\theta\ d\theta$$
$$= \frac{1}{3}\left[\sin^3\theta\right]_0^{\frac{\pi}{4}}$$
$$= \frac{1}{3}\left(\sin^3\ \frac{\pi}{4} - 0\right)$$
$$= \frac{1}{3}\left(\frac{1}{\sqrt{2}}\right)^3$$
$$= \frac{1}{6\sqrt{2}}$$
$$= \frac{\sqrt{2}}{12}$$

21. Calculus, EXT1 C2 2022 HSC 12c

$$y = x\tan^{-1}(x)$$

$$\frac{dy}{dx} = x \times \frac{1}{1+x^2} + \tan^{-1}(x)$$

When $x = 1$:

$$\frac{dy}{dx} = \frac{1}{2} + \tan^{-1}(1) = \frac{1}{2} + \frac{\pi}{4} = \frac{2+\pi}{4}$$

Equation of tangent $m = \frac{2+\pi}{4}$, through $\left(1, \frac{\pi}{4}\right)$:

$$y - \frac{\pi}{4} = \frac{2+\pi}{4}(x-1)$$

$$y = \left(\frac{2+\pi}{4}\right)x - \frac{2+\pi}{4} + \frac{\pi}{4}$$

$$y = \left(\frac{2+\pi}{4}\right)x - \frac{1}{2}$$

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22. Calculus, EXT1 C2 2016 HSC 11c

$$y = 3\tan^{-1}(2x)$$

$$\frac{dy}{dx} = \frac{3}{1+(2x)^2} \times 2$$

$$= \frac{6}{1+4x^2}$$

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23. Calculus, EXT1 C2 2009 HSC 1f

$$u = x^3 + 1$$

$$\frac{du}{dx} = 3x^2$$

$$du = 3x^2 \ dx$$

If $x = 2,\ u = 9$

$x = 0,\ u = 1$

$$\therefore \int_0^2 x^2 e^{x^3+1} \ dx$$

$$= \frac{1}{3}\int_0^2 e^{x^3+1} \cdot 3x^2 \ dx$$

$$= \frac{1}{3}\int_1^9 e^u \ du$$

$$= \frac{1}{3}\left[e^u\right]_1^9$$

$$= \frac{1}{3}\left(e^9 - e\right)$$

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24. Calculus, EXT1 C2 2007 HSC 1e

$$u = 25 - x^2$$

$$\frac{du}{dx} = -2x$$

$$du = -2x\ dx$$

$$\text{If } x = 4,\ u = 9$$

$$x = 3,\ u = 16$$

$$\therefore \int_3^4 \frac{2x}{\sqrt{25 - x^2}}\ dx$$

$$= -\int_{16}^{9} u^{-\frac{1}{2}} du$$

$$= -\left[\frac{1}{\frac{1}{2}} u^{\frac{1}{2}}\right]_{16}^{9}$$

$$= -\left[2\sqrt{u}\right]_{16}^{9}$$

$$= -\left[2\sqrt{9} - 2\sqrt{16}\right]$$

$$= -[6 - 8]$$

$$= 2$$

MARKER'S COMMENT: A "significant number" of students put the integral limits in the wrong order.

25. Calculus, EXT1 C2 2013 HSC 11b

$$\int \frac{1}{\sqrt{49 - 4x^2}}\ dx$$

$$= \int \frac{1}{2\sqrt{\frac{49}{4} - x^2}}\ dx$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{\left(\frac{7}{2}\right)^2 - x^2}}\ dx$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{2x}{7}\right) + c$$

26. Calculus, EXT1 C2 2013 HSC 11f

$$\text{Let } u = e^{3x}$$

$$\frac{du}{dx} = 3e^{3x}$$

$$\therefore dx = \frac{du}{3e^{3x}}$$

$$\text{When } x = \frac{1}{3},\ u = e^{3 \times \frac{1}{3}} = e$$

$$x = 0,\quad u = e^0 = 1$$

$$\therefore \int_0^{\frac{1}{3}} \frac{e^{3x}}{e^{6x} + 1}\ dx$$

$$= \int_1^e \frac{e^{3x}}{u^2 + 1} \times \frac{du}{3e^{3x}}$$

$$= \frac{1}{3}\int_1^e \frac{1}{u^2 + 1}\ du$$

$$= \frac{1}{3}\left[\tan^{-1} u\right]_1^e$$

$$= \frac{1}{3}\left[\tan^{-1} e - \tan^{-1} 1\right]$$

$$= \frac{1}{3}\left(\tan^{-1} e - \frac{\pi}{4}\right)$$

MARKER'S COMMENT: Many students did not calculate in radians and incorrectly got an answer of 8.2. *BE CAREFUL!* Note that converting your answer to 0.14 is also correct but not required.

27. Calculus, EXT1 C2 2014 HSC 11d

$$x = u^2 + 1$$
$$u^2 = x - 1$$
$$u = \sqrt{x-1}$$
$$\frac{du}{dx} = \frac{1}{2}(x-1)^{-\frac{1}{2}}$$
$$= \frac{1}{2\sqrt{x-1}}$$
$$\Rightarrow 2du = \frac{dx}{\sqrt{x-1}}$$

When $x = 5, \quad u = 2$

$x = 2, \quad u = 1$

$$\therefore \int_2^5 \frac{x}{\sqrt{x-1}}\ dx$$
$$= 2\int_1^2 u^2 + 1\ du$$
$$= 2\left[\frac{u^3}{3} + u\right]_1^2$$
$$= 2\left[\left(\frac{8}{3} + 2\right) - \left(\frac{1}{3} + 1\right)\right]$$
$$= \frac{20}{3}$$

---

28. Calculus, EXT1 C2 2018 HSC 12c

i. $f\prime(x) = \dfrac{1}{\sqrt{1-x^2}} + \left(-\dfrac{1}{\sqrt{1-x^2}}\right) = 0$

ii. Since $f\prime(x) = 0 \quad \Rightarrow f(x)$ is a constant.

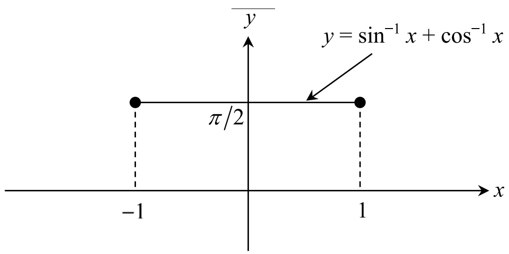
♦ *Mean mark 37%.*

Substituting $x = 1$ into the equation (any value works)

$$\sin^{-1}1 + \cos^{-1}1 = \frac{\pi}{2} + 0$$
$$= \frac{\pi}{2} \quad \ldots \text{as required}$$

iii. Domain restrictions require: $-1 < x < 1$

♦ *Mean mark 40%.*



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29. Calculus, EXT1 C2 2023 HSC 14a

Mean mark (i) 54%.

i. $f(x) = 2x + \ln x$

$f'(x) = 2 + \dfrac{1}{x}$

In domain $x > 0 \ \Rightarrow f^{-1}(x) > 0 \ (f(x)$ is monotonically increasing$)$

Since $f(x)$ is one-to-one, $f^{-1}(x)$ is a function.

♦♦ Mean mark (ii) 32%.

ii. $g(x) = f^{-1}(x)$

$f(g(x)) = x$

Differentiate both sides:

$g'(x)\ f'(g(x)) = 1$

$$g'(x) = \frac{1}{f'(g(x))}$$

$$g'(2) = \frac{1}{f'(g(2))}$$

$f(1) = 2 \times 1 + \ln 1 = 2$

$\Rightarrow g(2) = 1$ (by inverse definition)

$$\therefore g'(2) = \frac{1}{f'(1)} = \frac{1}{2 + \frac{1}{1}} = \frac{1}{3}$$

30. Calculus, EXT1 C2 2019 HSC 14c

i. $y_1 = \sin x$

$\dfrac{dy_1}{dx} = \cos x$

$y_2 = \sin(x - \alpha) + k$

$\dfrac{dy_2}{dx} = \cos(x - \alpha)$

At $x = x_0$, tangent is common

$\therefore \cos x_0 = \cos(x_0 - \alpha)$

♦ *Mean mark part (i) 47%.*

ii. x_0 is in 1st quadrant (given)

Using part (i):

$\cos x_0 = \cos(x_0 - \alpha) > 0$

$\Rightarrow x_0 - \alpha$ is in 4th quadrant $(0 < \alpha < \pi)$

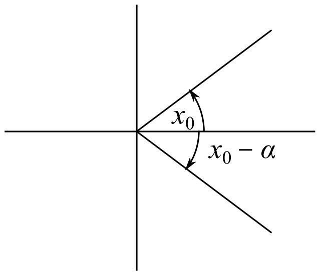
♦♦♦ *Mean mark part (ii) 19%.*

Since sin is positive in 1st quadrant and

negative in 4th quadrant

$\Rightarrow \sin x_0 = -\sin(x_0 - \alpha)$

iii.



When $x = x_0$,

$y_1 = \sin x_0$

$y_2 = \sin(x_0 - \alpha) + k$

$\sin x_0 = \sin(x_0 - \alpha) + k$

$= -\sin x_0 + k$

$k = = 2\ \sin x_0$

$$\text{Since}\quad \cos x_0 = \cos(x_0 - \alpha)$$
$$x_0 = -(x_0 - \alpha)$$
$$2x_0 = \alpha$$
$$x_0 = \frac{\alpha}{2}$$

$$\therefore k = 2\sin\frac{\alpha}{2}$$

♦♦ *Mean mark part (iii) 21%.*

31. Calculus, EXT1 C2 2020 HSC 13c

i. $f(x) = \tan\left(\cos^{-1}(x)\right)$

$$f'(x) = -\frac{1}{\sqrt{1-x^2}} \cdot \sec^2\left(\cos^{-1}(x)\right)$$
$$= -\frac{1}{\sqrt{1-x^2}} \cdot \frac{1}{\cos^2\left(\cos^{-1}(x)\right)}$$
$$= -\frac{1}{x^2\sqrt{1-x^2}}$$

♦ *Mean mark (i) 50%.*

$$g(x) = \left(1-x^2\right)^{\frac{1}{2}} \cdot x^{-1}$$
$$g'(x) = \frac{1}{2} \cdot -2x\left(1-x^2\right)^{-\frac{1}{2}} \cdot x^{-1} - \left(1-x^2\right)^{\frac{1}{2}} \cdot x^{-2}$$
$$= \frac{-x}{x\sqrt{1-x^2}} - \frac{\sqrt{1-x^2}}{x^2}$$
$$= \frac{-x^2 - \sqrt{1-x^2}\sqrt{1-x^2}}{x^2\sqrt{1-x^2}}$$
$$= \frac{-x^2 - \left(1-x^2\right)}{x^2\sqrt{1-x^2}}$$
$$= -\frac{1}{x^2\sqrt{1-x^2}}$$
$$= f'(x)$$

ii. $f'(x) = g'(x)$

$$\Rightarrow f(x) = g(x) + c$$

♦♦♦ *Mean mark (ii) 15%.*

Find c:

$$f(1) = \tan\left(\cos^{-1}1\right)$$
$$= \tan 0$$
$$= 0$$
$$g(1) = \frac{\sqrt{1-1}}{0} = 0$$
$$f(1) = g(1) + c$$
$$\therefore c = 0$$
$$\therefore f(x) = g(x)$$

32. Calculus, EXT1 C2 2021 HSC 14e

$g(x) = x^3 + 4x - 2$

$g\prime(x) = 3x^2 + 4$

$f(x) = xg^{-1}(x)$

$f\prime(x) = x \cdot \dfrac{d}{dx} g^{-1}(x) + g^{-1}(x)$

♦♦♦ *Mean mark 10%.*

COMMENT: The reciprocal relationship of gradients between $g(x)$ and $g^{-1}(x)$ is critical here.

$g(x)$ passes through $(1, 3)$

$\Rightarrow g^{-1}(x)$ passes through $(3, 1)$

$g\prime(1) = 3 + 4 = 7$

$\Rightarrow \dfrac{d}{dx} g^{-1}(3) = \dfrac{1}{\frac{d}{dy} g(y)} = \dfrac{1}{g\prime(1)} = \dfrac{1}{7}$

$$\begin{aligned} \therefore f\prime(x)\ |_{x=3} &= 3 \cdot \frac{1}{7} + 1 \\ &= \frac{10}{7} \end{aligned}$$