MATHEMATICS EXTENSION 2

SmarterEd Maths & Sciences

COMPREHENSIVE REVISION SERIES 2024

Thank you for subscribing to SmarterMaths Teacher Edition (Silver) in 2024.

Key features of the Extension 2 “2024 HSC Comprehensive Revision Series” include:

- ~16 hours of cherry-picked HSC revision questions by topic
- Targeted at motivated students aiming for a Band 5 or 6 equivalent result
- Weighting toward more difficult examples
- Mark allocations given to each topic generally reflect its historical (new syllabus) HSC exam allocation.
- ***Attempt, carefully review and annotate*** this revision set in Term 3
- This question set provides the foundation of a concise and high-quality revision resource for the run into the HSC exam.
- This resource should be used to complement (not replace) the critical final stretch preparation for every student – completing full practice papers in exam conditions.

Our analysis on each topic, the common question types, past areas of difficulty and recent HSC trends all combine to create this revision set that ensures students cover a wide cross-section of the key areas.

IMPORTANT: If students have been exposed to questions in these worksheets during the year, we say great. Many top performing students attest to the benefits of doing quality questions 2-3 times before the HSC. This type of revision set is aimed at creating confidence and *speed through the exam*, with cherry picked questions that cover all important elements of revision while avoiding low percentage rabbit hole excursions.

HSC Final Study: P1 Nature of Proof (2 of 2) (~15.0% historical contribution)

Key Areas addressed by this worksheet

Converse, Contradiction and Contrapositive Proof (6.4%)

- *Converse, Contradiction and Contrapositive Proof* has been examined at least once in the longer answer section of every new syllabus exam to date (including twice in 2020), as well as contributing at least two multiple choice questions to each exam.
- *Contrapositive Proof* has easily been the most common question type. Notably absent in the 2023 exam, this area was examined twice each year between 2020-2022 across a broad spectrum of difficulty levels.
- While most were low-mid band and well handled, *2020 Ext2 15a* and *2021 Ext2 9 MC* both produced sub-50% mean marks and are important revision questions.
- *Converse Proofs* have also appeared in each new syllabus exam, twice within the longer answer section. *2022 Ext2 7 MC* and *2020 Ext2 15a* (note this example combined converse and contrapositive proofs) both caused problems and are included in this revision worksheet.
- *Contradiction* can have broad applications for proof (see *2020 Ext2 7 MC* and *2021 Ext2 5 MC*), with its most important being for proof of irrationality, last examined in *2023 Ext2 12a* and *2020 Ext2 14d*.

> *"After just 1 term, I am noticing that my students are in a better position to tackle exams by basing our revision around SmarterMaths worksheets."*
>
> *~ Adrian Kruse, MacArthur Anglican*

EXTENSION 2

Stage 6

2024 Comprehensive Revision Series

PROOF

P1 Nature of Proof

- Converse, Contradiction and Contrapositive Proof

Exam Equivalent Time: 45 minutes *(based on allocation of 1.5 minutes per mark)*

Questions

1. Proof, EXT2 P1 2020 HSC 7 MC

Consider the proposition:

'If $2^n - 1$ is not prime, then n is not prime'.

Given that each of the following statements is true, which statement disproves the proposition?

A. $2^5 - 1$ is prime

B. $2^6 - 1$ is divisible by 9

C. $2^7 - 1$ is prime

D. $2^{11} - 1$ is divisible by 23

---

2. Proof, EXT2 P1 2023 HSC 2 MC

Consider the following statement.

'If an animal is a herbivore, then it does not eat meat.'

Which of the following is the converse of this statement?

A. If an animal is a herbivore, then it eats meat.

B. If an animal is not a herbivore, then it eats meat.

C. If an animal eats meat, then it is not a herbivore.

D. If an animal does not eat meat, then it is a herbivore.

---

3. Proof, EXT2 P1 2022 HSC 3 MC

Let A, B, P be three points in three-dimensional space with $A \neq B$.

Consider the following statement.

If P is on the line AB, then there exists a real number λ such that $\overrightarrow{AP} = \lambda\overrightarrow{AB}$.

Which of the following is the contrapositive of this statement?

A. If for all real numbers λ, $\overrightarrow{AP} = \lambda\overrightarrow{AB}$, then P is on the line AB.

B. If for all real numbers λ, $\overrightarrow{AP} \neq \lambda\overrightarrow{AB}$, then P is not on the line AB.

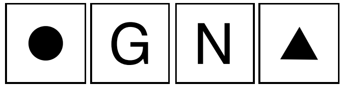
C. If there exists a real number λ such that $\overrightarrow{AP} = \lambda\overrightarrow{AB}$, then P is on the line AB.

D. If there exists a real number λ such that $\overrightarrow{AP} \neq \lambda\overrightarrow{AB}$, then P is not on the line AB.

---

4. Proof, EXT2 P1 SM-Bank 5 MC

Four cards are placed on a table with a letter on one face and a shape on the other.



You are given the rule: "if N is on a card then a circle is on the other side."

Which cards need to be turned over to check if this rule holds?

A. N and G

B. G and triangle

C. circle and N

D. N and triangle

---

5. Proof, EXT2 P1 2021 HSC 5 MC

Which of the following statements is FALSE?

A. $\forall a, b \in \mathbb{R}, \quad a < b \Rightarrow a^3 < b^3$

B. $\forall a, b \in \mathbb{R}, \quad a < b \Rightarrow e^{-a} > e^{-b}$

C. $\forall a, b \in (0, +\infty), \quad a < b \Rightarrow \ln a < \ln b$

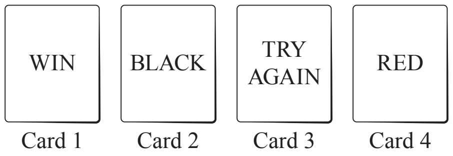
D. $\forall a, b \in \mathbb{R}, \text{with } a, b \neq 0, \quad a < b \Rightarrow \dfrac{1}{a} > \dfrac{1}{b}$

---

6. Proof, EXT2 P1 2021 HSC 9 MC

Four cards have either RED or BLACK on one side and either WIN or TRY AGAIN on the other side.

Sam places the four cards on the table as shown below.



A statement is made: ‘If a card is RED, then it has WIN written on the other side’.

Sam wants to check if the statement is true by turning over the minimum number of cards.

Which cards should Sam turn over?

A. 1 and 4

B. 3 and 4

C. 1. 2 and 4

D. 1, 3 and 4

15/08/2024, 15:50 SmarterMaths Client App

7. Proof, EXT2 P1 2022 HSC 7 MC

Consider the statement P.

P: For all integers $n \geq 1$, if n is a prime number then $\dfrac{n(n+1)}{2}$ is a prime number.

Which of the following is true about this statement and its converse?

A. The statement P and its converse are both true.

B. The statement P and its converse are both false.

C. The statement P is true and its converse is false.

D. The statement P is false and its converse is true.

---

8. Proof, EXT2 P1 2020 HSC 8 MC

Consider the statement:

'If n is even, then if n is a multiple of 3, then n is a multiple of 6'.

Which of the following is the negation of this statement?

A. n is odd and n is not a multiple of 3 or 6.

B. n is even and n is a multiple of 3 but not a multiple of 6.

C. If n is even, then n is not a multiple of 3 and n is not a multiple of 6.

D. If n is odd, then if n is not a multiple of 3 then n is not a multiple of 6.

---

9. Proof, EXT2 P1 2021 HSC 12b

Consider Statement A.

Statement A: 'If n^2 is even, then n is even.'

i. What is the converse of Statement A?. *(1 mark)*

ii. Show that the converse of Statement A is true. *(1 mark)*

10. Proof, EXT2 P1 SM-Bank 15

Prove $\sqrt{5}+\sqrt{3}>\sqrt{14}$ by contradiction. *(2 marks)*

11. Proof, EXT2 P1 2020 HSC 14d

Prove that for any integer $n>1, \log_n(n+1)$ is irrational. *(3 marks)*

12. Proof, EXT2 P1 2022 HSC 13a

Prove that for all integers n with $n \geq 3$, if 2^n-1 is prime, then n cannot be even. *(3 marks)*

13. Proof, EXT2 P1 SM-Bank 4

Prove that $\sqrt[3]{2}$ is irrational. *(3 marks)*

---

14. Proof, EXT2 P1 SM-Bank 9

If n is a positive integer,

prove $\sqrt{10n+2}$ is always irrational. *(3 marks)*

15/08/2024, 15:50 SmarterMaths Client App

15. Proof, EXT2 P1 2020 HSC 15a

In the set of integers, let P be the proposition:

'If $k+1$ is divisible by 3, then k^3+1 divisible by 3.'

i. Prove that the proposition P is true. *(2 marks)*

ii. Write down the contrapositive of the proposition P. *(1 mark)*

iii. Write down the converse of the proposition P and state, with reasons, whether this converse is true or false. *(3 marks)*

Worked Solutions

Worked Solutions

1. Proof, EXT2 P1 2020 HSC 7 MC

Strategy 1 – Contradiction

Consider option D,

Since $2^{11}-1$ is divisible by 23, it is NOT prime.

The proposition states that 11 is not prime which is false.

$\therefore 2^{11}-1$ is divisible by 23, disproves the proposition.

Strategy 2 – Contrapositive

The proposition is conditional

$X \Rightarrow Y$

Logically equivalent contrapositive statement

$\neg\, Y \Rightarrow \neg\, X$

i.e. If n is prime $\Rightarrow$ 2^n-1 is prime.

Consider D:

$n = 11$ (prime)

$2^{11}-1$ is divisible by 23 (not prime)

$\therefore$ Contrapositive statement is false and disproves the proposition.

$\Rightarrow D$

---

2. Proof, EXT2 P1 2023 HSC 2 MC

Statement: $P \Rightarrow Q$

Converse of statement: $Q \Rightarrow P$

$\Rightarrow D$

---

3. Proof, EXT2 P1 2022 HSC 3 MC

Statement:

If P is on AB $\Rightarrow$ $\exists\lambda$ such that $\overrightarrow{AP} = \lambda\overrightarrow{AB}$

Contrapositive statement:

If $\neg\ \exists\lambda$ such that $\overrightarrow{AP} = \lambda\overrightarrow{AB}$ $\Rightarrow$ P is not on AB

In other words . . .

If for all real numbers λ, $\overrightarrow{AP} \neq \lambda\overrightarrow{AB}$, then P is not on the line AB.

$\Rightarrow B$

---

4. Proof, EXT2 P1 SM-Bank 5 MC

Solution 1

Logically equivalent statements are:

$N \Rightarrow$ circle

$\neg$ circle $\Rightarrow \neg N$

To confirm rule is not broken,

N must be turned

Triangle (not circle) must be turned – only other shape not a circle.

$\Rightarrow D$

Solution 2

Consider the flip side of each card.

If circle has N on the other side (or not)−tells us nothing.

If G has a circle on the other side (or not)−tells us nothing.

If N doesn't have a circle on other side−rule broken.

If triangle has an N on other side−rule broken.

$\therefore$ Need to turn N and triangle

---

5. Proof, EXT2 P1 2021 HSC 5 MC

By contradiction:

Consider D

♦ *Mean mark 50%.*

Let $a = -1$ and $b = 1$,

$a < b \;\rightarrow\; -1 < 1$ (TRUE)

$\dfrac{1}{a} > \dfrac{1}{b} \;\rightarrow\; -1 > 1$ (FALSE)

$\Rightarrow$ D is false

---

6. Proof, EXT2 P1 2021 HSC 9 MC

Logically equivalent statements are:

Red $\Rightarrow$ Win ... (1)

$\neg$ Win $\Rightarrow$ $\neg$ Red ... (2)

To confirm statement is true

Card 1 – no need to turn

♦ *Mean mark 48%.*

Card 2 – no need to turn

Card 3 – turn to confirm (2)

Card 4 – turn to confirm (1)

$\Rightarrow B$

---

7. Proof, EXT2 P1 2022 HSC 7 MC

Statement: $\forall n \in \mathbb{Z}^+$, if n is prime $\Rightarrow$ $\dfrac{n(n+1)}{2}$ is prime.

Converse: $\forall n \in \mathbb{Z}^+$, if $\dfrac{n(n+1)}{2}$ is prime $\Rightarrow$ n is prime.

Consider prime $n = 3$:

♦♦ Mean mark 37%.

$\dfrac{n(n+1)}{2} = \dfrac{3 \times 4}{2} = 6$ (not prime $\rightarrow$ Statement is false)

Consider $\dfrac{n(n+1)}{2}$:

$\dfrac{n(n+1)}{2}$ is a prime $\Leftrightarrow$ $n = 2$ (prime)

$\therefore$ Converse is true

$\Rightarrow D$

---

8. Proof, EXT2 P1 2020 HSC 8 MC

Proposition: If $X \Rightarrow Y$

X is a compound statement

♦♦ *Mean mark part 33%.*

"If n is even and a multiple of 3."

Y states "n is a multiple of 6."

Negation if X but $\neg\, Y$.

$\Rightarrow$ B

---

9. Proof, EXT2 P1 2021 HSC 12b

i. Converse

If n is even, then n^2 is even.

ii. If n is even:

$$n = 2p,\ p \in \mathbb{Z}$$
$$n^2 = (2p)^2$$
$$= 4p^2$$
$$= 2\left(2p^2\right)$$
$$= 2q,\ q \in \mathbb{Z}$$

$\therefore$ If n is even, then n^2 is even.

---

10. Proof, EXT2 P1 SM-Bank 15

Proof by contradiction:

Assume $\sqrt{5}+\sqrt{3} \le \sqrt{14}$

$$\left(\sqrt{5}+\sqrt{3}\right)^2 \le \left(\sqrt{14}\right)^2$$
$$5 + 2\sqrt{15} + 3 \le 14$$
$$2\sqrt{15} \le 6$$
$$\sqrt{15} \le 3$$
$$15 \le 9 \quad \text{(incorrect)}$$

$\therefore$ By contradiction, $\sqrt{5}+\sqrt{3} > \sqrt{14}$

---

11. Proof, EXT2 P1 2020 HSC 14d

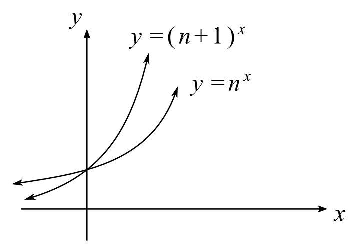
Proof by contradiction:

Assume $\log_n(n+1)$ is rational

$\therefore\ \log_n(n+1) = \dfrac{p}{q}$ where $p, q \in \mathbb{Z}$ with no common factor except 1

$$n^{\frac{p}{q}} = n+1$$
$$n^p = (n+1)^q$$

Strategy 1



$n^p = (n+1)^q$ when $p = q = 0$ only

$q \neq 0$

$\therefore$ By contradiction, $\log_n(n+1)$ is irrational.

Strategy 2

$n^p = (n+1)^q$

If n is odd, LHS is odd and RHS is even.

If n is even LHS is even and RHS is odd.

Statement is true for $p = q = 0$, but $q \neq 0$

$\therefore$ By contradiction, $\log_n(n+1)$ is irrational.

---

12. Proof, EXT2 P1 2022 HSC 13a

Contrapositive statement:

If n is even, $2^n - 1$ is NOT prime.

Let $n = 2k$, $(k \in \mathbb{Z}$ and $k \geq 2)$

$$\begin{aligned} 2^n - 1 &= 2^{2k} - 1 \\ &= \left(2^k\right)^2 - 1 \\ &= \left(2^k - 1\right)\left(2^k + 1\right) \end{aligned}$$

Since $k \geq 2 \Rightarrow 2^k - 1 \geq 3$ and $2^k + 1 \geq 5$

$\therefore 2^n - 1$ is not prime if n is even, as it has two non-trivial integer factors.

$\therefore$ By contrapositive statement, if $2^n - 1$ is prime, n cannot be even.

13. Proof, EXT2 P1 SM-Bank 4

Proof by contradiction:

Assume that $\sqrt[3]{2}$ is rational.

$\therefore \sqrt[3]{2} = \dfrac{p}{q}$ where $p, q \in \mathbb{Z}$ with no common factor except 1

$$2 = \frac{p^3}{q^3}$$

$$2q^3 = p^3 \ldots (1)$$

p^3 is even $\Rightarrow$ p is even

(i.e.) $\exists k$, $k \in \mathbb{Z}$ such that $p = 2k$

Substitute $q = 2k$ into (1)

$$\begin{aligned} 2q^3 &= (2k)^3 \\ 2q^3 &= 8k^3 \\ q^3 &= 4k^3 \end{aligned}$$

q^3 is even $\Rightarrow$ q is even

$\therefore p$ and q have a common factor of 2

$\Rightarrow$ Contradiction

$\therefore \sqrt[3]{2}$ is irrational.

14. Proof, EXT2 P1 SM-Bank 9

Proof by contradiction:

Assume that $\sqrt{10n+2}$ is rational.

$\therefore \sqrt{10n+2} = \dfrac{p}{q}$ where $p, q \in \mathbb{Z}$ with no common factor except 1

$$10n + 2 = \frac{p^2}{q^2}$$

$2q^2(5n+1) = p^2 \ldots (1)$

p^2 is even $\Rightarrow$ p is even

(i.e.) $\exists k$, $k \in \mathbb{Z}$ such that $p = 2k$

Substitute $p = 2k$ into (1)

$$2q^2(5n+1) = (2k)^2$$
$$q^2(5n+1) = 2k^2$$

$q^2(5n+1)$ is even since $\Rightarrow$ $2k^2$ is even

$\Rightarrow q^2$ is even since $5n+1$ can be odd or even.

$\Rightarrow q$ is even

$\Rightarrow$ Contradiction: p and q have a common factor of 2

$\therefore$ $\sqrt{10n+2}$ is irrational.

15. Proof, EXT2 P1 2020 HSC 15a

i. Let $k+1 = 3N$, $N \in Z$

$\Rightarrow$ $k = 3N - 1$

$$\begin{aligned} k^3 + 1 &= (3N-1)^3 + 1 \\ &= (3N)^3 + 3(3N)^2(-1) + 3(3N)(-1)^2 + (-1)^3 + 1 \\ &= 27N^3 - 27N^2 + 9N - 1 + 1 \\ &= 3\left(9N^3 - 9N^2 + 3N\right) \\ &= 3Q\ ,\ Q \in Z \end{aligned}$$

$\therefore$ If $k+1$ is divisible by 3, then k^3+1 is divisible by 3.

ii. Contrapositive

If k^3+1 is not divisible by 3, then $k+1$ is not divisible by 3.

iii. Converse:

♦♦ *Mean mark part (iii) 36%.*

If k^3+1 is divisible by 3, then $k+1$ is divisible by 3.

Contrapositive of converse:

If $k+1$ is not divisible by 3, then k^3+1 is not divisible by 3.

i.e. $k+1$ is not divisible by 3 when $k+1 = 3Q+1$ or $k+1 = 3Q+2$, where $Q \in Z$

If $k+1 = 3Q+1$ $\Rightarrow$ $k = 3Q$

$$\begin{aligned} k^3 + 1 &= (3Q)^3 + 1 \\ &= 27Q^3 + 1 \\ &= 3\left(9Q^3\right) + 1 \\ &= 3M + 1 \quad (\text{not divisible by 3, } M \in Z) \end{aligned}$$

If $k+1 = 3Q+2$ $\Rightarrow$ $k = 3Q+1$

$$\begin{aligned} k^3 + 1 &= (3Q+1)^3 + 1 \\ &= (3Q)^3 + 3(3Q)^2 + 3(3Q) + 1 + 1 \\ &= 27Q^3 + 27Q^2 + 9Q + 2 \\ &= 3\left(9Q^3 + 9Q^2 + 3Q\right) + 2 \end{aligned}$$

$= 3M + 2$ (not divisible by 3, $M \in Z$)

$\therefore$ By contrapositive, if $k^3 + 1$ is divisible by 3, $k + 1$ is divisible by 3.